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Irreducible representations of Leavitt path algebras

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Abstract. We construct some irreducible representations of the Leavitt path algebra of an arbitrary quiver. The constructed representations are associated to certain algebraic branching systems. For a row-finite quiver, we classify algebraic branching systems, to which irreducible representations of the Leavitt path algebra are associated. For a certain quiver, we obtain a faithful completely reducible representation of the Leavitt path algebra. The twisted representations of the constructed ones under the scaling action are studied.

Keywords. Quiver, Leavitt path algebra, irreducible representation, left-infinite path, algebraic branching system.

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1 Introduction

Let k be a field and let Q be an arbitrary quiver. The notion of the path algebra kQ of Q is well known in representation theory ([11]). Unlike this, the Leavitt path algebra $L_k(Q)$ of Q with coefficients in k is relatively new, which is introduced in [1, 7, 8]. Leavitt path algebras generalize the important algebras studied by Leavitt in [19, 20], and are algebraic analogues of the Cuntz–Krieger C^*-algebras $C^*(Q)$ ([16, 23]). Recent research indicates that the Leavitt path algebra of a quiver captures certain homological properties of both the path algebra and its Koszul dual; see [6, 14, 24].

The representation theory of the Leavitt path algebra $L_k(Q)$ is studied in the papers [5, 6, 17]. In [6], the authors prove that the category of finitely presented $L_k(Q)$-modules is equivalent to a quotient category of the corresponding category of kQ-modules. This result is extended in [24] via a completely different method. Using the notion of algebraic branching system, a construction of $L_k(Q)$-modules is given in [17]. Moreover, some sufficient conditions are given to guarantee the faithfulness of the constructed modules.

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We are interested in simple modules, or equivalently, irreducible representations of Leavitt path algebras. Recall that irreducible representations that can be embedded in the Leavitt path algebra itself are just minimal left ideals. These representations are classified in [9, 10]. This classification plays an important role in the study of the socle series of Leavitt path algebras; see [4].

In this paper, we construct some irreducible representations of the Leavitt path algebra $L_k(Q)$ of an arbitrary quiver Q. More precisely, we prove the following theorem.

Recall that $L_k(Q)$ is generated by e_i, α and α^* for all vertices i and arrows α in the quiver Q. By a left-infinite path, we mean an infinite path which is unbounded on the left. For a left-infinite path p and an arrow α, we denote by $p\alpha$ their concatenation if p starts at the terminating vertex of α. We denote the action of an algebra on modules by ".".

Theorem. *Let Q be an arbitrary quiver. Let $\mathcal{F}$ be the linear span of all left-infinite paths in Q and $\mathcal{N}$ the linear span of all finite paths in Q that terminate at a sink. Then the following statements hold.*

(1) *$\mathcal{F} \oplus \mathcal{N}$ is a left $L_k(Q)$-module by $e_i.p = p$ if p starts at i and $e_i.p = 0$ otherwise, $\alpha.p = p'$ if $p = p'\alpha$ and $\alpha.p = 0$ otherwise, $\alpha^*.p = p\alpha$ if p starts at the terminating vertex of α and $\alpha^*.p = 0$ otherwise.*

(2) *The representation $\mathcal{F} \oplus \mathcal{N}$ is a direct sum of irreducible representations, each of which occurs with multiplicity one.*

The construction of the modules is inspired by a construction of representations of Cuntz algebras in [22]. The irreducible subrepresentations contained in $\mathcal{F}$ relate to the point modules studied in [24, 25], while the latter plays an important role in non-commutative algebraic geometry. The irreducible subrepresentations contained in $\mathcal{N}$ are isomorphic to minimal left ideals of $L_k(Q)$ that is generated by idempotents corresponding to sinks of the quiver Q; these representations are known, at least for countable quivers ([9, 10]).

The paper is structured as follows. We recall some basic notions and introduce some terminology in Section 2. The main construction is given in Section 3, where the above theorem is contained in Theorems 3.3 and 3.7. In Section 4, we draw some consequences from the constructed representations. Based on results in [10], we point out that for a countable quiver, the constructed irreducible representations contain all minimal left ideals of the Leavitt path algebra; see Proposition 4.3. We prove the faithfulness of the representation $\mathcal{F} \oplus \mathcal{N}$ for certain quivers; see Proposition 4.4. We relate irreducible subrepresentations of $\mathcal{F}$ to point modules; see Proposition 4.9. Section 5 is devoted to relating the constructed representations to algebraic branching systems in [17]. For a row-finite quiver, we classify algebraic branching systems whose associated representations are irreducible. It turns out that irreducible representations associated to algebraic branching systems are necessarily isomorphic to the ones constructed in Section 3; see Theorem 5.4. In the final section, we study the twisted representations of the constructed irreducible representations under the scaling action. This allows us to obtain new irreducible representations and prove the faithfulness of some completely reducible representation; see Theorem 6.2 and Proposition 6.3.

2 Preliminaries

We recall basic notions related to quivers and Leavitt path algebras, and introduce some terminology for later use. The references for quivers are [11, Chapter III] and [1], and for Leavitt path algebras are [1, 2, 8, 26].

2.1 Quivers and left-infinite paths

Recall that a *quiver* $Q = (Q_0, Q_1; s, t)$ consists of a set Q_0 of vertices, a set Q_1 of arrows and two maps $s, t\colon Q_1 \to Q_0$, which assign an arrow α to its starting and terminating vertices $s(\alpha)$ and $t(\alpha)$, respectively. A quiver is also called a directed graph. A vertex where there is no arrow starting is called a *sink*, and a vertex where there are infinitely many arrows starting is called an *infinite emitter*. A vertex is *regular* if it is neither a sink nor an infinite emitter. The quiver Q is *regular* (resp. *row-finite*) provided that each vertex is regular (resp. not an infinite emitter).

A (finite) *path* in the quiver Q is a sequence $p = \alpha_n \cdots \alpha_2\alpha_1$ of arrows with $t(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq n-1$; in this case, the path p is said to have length n, denoted by $l(p) = n$. We denote $s(p) = s(\alpha_1)$ and $t(p) = t(\alpha_n)$. We identify an arrow with a path of length one, and associate to each vertex i a trivial path e_i of length zero. A nontrivial path p with the same starting and terminating vertex is an *oriented cycle*. An oriented cycle of length one is called a *loop*.

Let k be a field. We denote by Q_n the set of paths of length n, and by kQ_n the vector space over k with basis Q_n. Here, we identify a vertex i with the corresponding trivial path e_i. The *path algebra* is defined as $kQ = \bigoplus_{n\geq 0} kQ_n$, whose multiplication is given as follows: for two paths p and q, if $s(p) = t(q)$, then the product pq is the concatenation of paths; otherwise, set the product pq to be zero. We write the concatenation of paths from the right to the left.

The path algebra kQ is naturally $\mathbb{N}$-graded. Observe that for a vertex i and a path p, $pe_i = \delta_{i,s(p)}p$ and $e_i p = \delta_{i,t(p)}p$. Here, δ is the Kronecker symbol. In particular, $\{e_i \mid i \in Q_0\}$ is a set of pairwise orthogonal idempotents in kQ. Observe that the k-algebra kQ is not necessarily unital unless Q has finitely many vertices.

We need infinite paths in a quiver. A *left-infinite path* in Q is an infinite sequence $p = \cdots \alpha_n \cdots \alpha_2\alpha_1$ of arrows with $t(\alpha_i) = s(\alpha_{i+1})$ for all $i \geq 1$. We set $s(p) = s(\alpha_1)$. Denote by Q_∞ the set of left-infinite paths in Q. For example, for an oriented cycle q, we have a left-infinite path $q^\infty = \cdots q \cdots qq$; such a left-infinite path is said to be *cyclic*. We remark that the set Q_∞ endowed with the product topology plays an important role in symbolic dynamics ([21]).

For a left-infinite path p and $n \geq 1$, denote by

$$\tau_{\leq n}(p) = \alpha_n \cdots \alpha_2\alpha_1 \quad \text{and} \quad \tau_{>n}(p) = \cdots \alpha_{n+2}\alpha_{n+1}$$

the two truncations. Observe that $\tau_{\leq n}(p)$ lies in Q_n and that $\tau_{>n}(p)$ is a left-infinite path. Hence, a left-infinite path p is cyclic if and only if there exists some $n \geq 1$ such that $p = \tau_{>n}(p)$. We set $\tau_{\leq 0}(p) = e_{s(p)}$ and $\tau_{>0}(p) = p$.

Two left-infinite paths p and q are *tail-equivalent*, denoted by $p \sim q$, provided that there exist n and m such that $\tau_{>n}(p) = \tau_{>m}(q)$; compare [25, Section 1.4]. This is an equivalence relation on Q_∞. We denote by $\widetilde{Q}_\infty$ the set of tail-equivalence classes, and for a path p denote the corresponding class by $[p]$.

A left-infinite path p is *rational* provided that there exists $n \geq 1$ such that $p \sim \tau_{>n}(p)$. This is equivalent to the condition that p is tail-equivalent to a cyclic path. In this case, $p \sim q^\infty$ for a *simple* oriented cycle q. Here, an oriented cycle is simple if it is not a power of a shorter oriented cycle. Otherwise, the path p is called *irrational*. This is equivalent to the condition that for each pair (n, m) of distinct natural numbers, we have $\tau_{>n}(p) \neq \tau_{>m}(p)$.

If a left-infinite path p is rational (resp. irrational), then the corresponding class $[p]$ is called a *rational class* (resp. an *irrational class*); such classes form a subset $\widetilde{Q}_\infty^{\mathrm{rat}}$ (resp. $\widetilde{Q}_\infty^{\mathrm{irr}}$) of $\widetilde{Q}_\infty$. Then we have a disjoint union $\widetilde{Q}_\infty = \widetilde{Q}_\infty^{\mathrm{rat}} \cup \widetilde{Q}_\infty^{\mathrm{irr}}$.

2.2 Leavitt path algebras

Let Q be a quiver and k a field. Consider the set of formal symbols $\{\alpha^* \mid \alpha \in Q_1\}$. The *Leavitt path algebra* $L_k(Q)$ of Q with coefficients in k is a k-algebra given by generators $\{e_i, \alpha, \alpha^* \mid i \in Q_0, \alpha \in Q_1\}$ subject to the following relations:

(0) $e_i e_j = \delta_{ij} e_i$ for all $i \in Q_0$,

(1) $e_{t(\alpha)}\alpha = \alpha = \alpha e_{s(\alpha)}$ for all $\alpha \in Q_1$,

(2) $e_{s(\alpha)}\alpha^* = \alpha^* = \alpha^* e_{t(\alpha)}$ for all $\alpha \in Q_1$,

(3) $\alpha\beta^* = \delta_{\alpha,\beta} e_{t(\alpha)}$ for all $\alpha, \beta \in Q_1$,

(4) $\sum_{\{\alpha\in Q_1 | s(\alpha)=i\}} \alpha^*\alpha = e_i$ for all regular vertices $i \in Q_0$.

The relations (3) and (4) are called the *Cuntz–Krieger relations*. Here, we emphasize that k-algebras are not required to be unital.

Observe that $L_k(Q)$ is naturally $\mathbb{Z}$-graded such that $\deg e_i = 0$, $\deg \alpha = 1$ and $\deg \alpha^* = -1$. There is a natural graded algebra homomorphism $\iota\colon kQ \to L_k(Q)$ such that $\iota(e_i) = e_i$ and $\iota(\alpha) = \alpha$. Here, we abuse notation: for a path $p \in kQ$ we denote its image $\iota(p)$ still by p. The algebra homomorphism ι is injective; see [18, Lemma 1.6] or Proposition 4.1.

For a path $p = \alpha_n \cdots \alpha_2\alpha_1$, we set $p^* = \alpha_1^*\alpha_2^* \cdots \alpha_n^*$ in $L_k(Q)$. By convention, $e_i^* = e_i$ for $i \in Q_0$. Indeed, there is an algebra anti-automorphism

$$(-)^*\colon L_k(Q) \to L_k(Q)$$

with the property that $(e_i)^* = e_i$, $(\alpha)^* = \alpha^*$ and $(\alpha^*)^* = \alpha$ for all vertices i and arrows α of the quiver Q.

The following fact is immediate from relation (3). Observe that for finite paths p, q in Q, $p^*q = 0$ if $t(q) \neq t(p)$.

Lemma 2.1 ([26, Lemma 3.1]). *Let p, q, γ and η be finite paths in Q such that $t(p) = t(q)$ and $t(\gamma) = t(\eta)$. Then in $L_k(Q)$ we have*

$$(p^*q)(\gamma^*\eta) = \begin{cases} (\gamma' p)^*\eta & \text{if } \gamma = \gamma' q, \\ p^*\eta & \text{if } q = \gamma, \\ p^*(q'\eta) & \text{if } q = q'\gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Here, γ' and q' are some nontrivial paths in Q.

We have the following immediate consequence; see [1, Lemma 1.5] or [26, Corollary 3.2].

Corollary 2.2. *The Leavitt path algebra $L_k(Q)$ is spanned by the following set:*

$$\{p^*q \mid p, q \text{ are finite paths in } Q \text{ with } t(p) = t(q)\}.$$

By Corollary 2.2, a nonzero element u in $L_k(Q)$ can be written in its *normal form*

$$u = \sum_{i=1}^{l} \lambda_i p_i^* q_i, \tag{2.1}$$

where $l \geq 1$, each $\lambda_i \in k$ is nonzero, and the p_i and q_i are paths in Q with $t(p_i) = t(q_i)$. We require in addition that the pairs (p_i, q_i) are pairwise distinct. The normal form in general is not unique because of relation (4).

Inspired by the paragraphs following [1, Lemma 1.7], we define a numerical invariant $\kappa(u)$ of u as the smallest natural number n_0 such that in one of its normal forms $u = \sum_{i=1}^{l} \lambda_i p_i^* q_i$, we have $l(p_i) \leq n_0$ for all i. For example, $\kappa(u) = 0$ if and only if u can be written as $u = \sum_{i=1}^{l} \lambda_i q_i$ for some paths q_i, if and only if u lies in the image of $\iota: kQ \to L_k(Q)$; compare [26, Definition 3.3].

The Leavitt path algebra $L_k(Q)$ in general is not unital. However, note that the set $\{e_i \mid i \in Q_0\}$ of pairwise orthogonal idempotents is a set of *local units* in the following sense: for a nonzero element $u = \sum_{i=1}^{l} \lambda_i p_i^* q_i$ in its normal form, set

$$x = \sum_{\{j \in Q_0 \mid j = s(p_i) \text{ for some } i\}} e_j \quad \text{and} \quad y = \sum_{\{j \in Q_0 \mid j = s(q_i) \text{ for some } i\}} e_j,$$

then we have $u = xuy$. In particular, there exists some $j \in Q_0$ such that $e_j u \neq 0$. For details, we refer to [1, Lemma 1.6] or [26, Section 3.2].

3 The construction of irreducible representations

In this section, we construct two classes of irreducible representations of Leavitt path algebras, and show that they are pairwise non-isomorphic.

3.1 The representation $\mathcal{F}$

Let k be a field and Q be a quiver. We denote by $\mathcal{F}$ the vector space over k with a basis given by the set Q_∞ of left-infinite paths in Q. For each tail-equivalence class $[p]$ in $\widetilde{Q}_\infty$, denote by $\mathcal{F}_{[p]}$ the subspace of $\mathcal{F}$ spanned by the set $\{q \mid q \in [p]\}$. Then we have

$$\mathcal{F} = \bigoplus_{[p] \in \widetilde{Q}_\infty} \mathcal{F}_{[p]}.$$

We will construct a representation of the Leavitt path algebra $L_k(Q)$ on $\mathcal{F}$. We point out that our construction is inspired by a construction in the proof of [22, Theorem II].

For each vertex $i \in Q_0$, define a linear map $P_i: \mathcal{F} \to \mathcal{F}$ such that

$$P_i(p) = \delta_{i,s(p)} p$$

for all $p \in Q_\infty$.

For each arrow $\alpha \in Q_1$, define a linear map $S_\alpha: \mathcal{F} \to \mathcal{F}$ such that

$$S_\alpha(p) = \delta_{\alpha,\alpha_1} \tau_{>1}(p)$$

for $p = \cdots \alpha_2 \alpha_1 \in Q_\infty$. We define another linear map $S_\alpha^*: \mathcal{F} \to \mathcal{F}$ such that

$$S_\alpha^*(p) = \delta_{t(\alpha),s(p)}\ p\alpha = \delta_{t(\alpha),s(\alpha_1)}\ p\alpha.$$

Here, we recall by definition that $s(p) = s(\alpha_1)$.

Proposition 3.1. *There is an algebra homomorphism $\rho: L_k(Q) \to \mathrm{End}_k(\mathcal{F})$ such that $\rho(e_i) = P_i$, $\rho(\alpha) = S_\alpha$ and $\rho(\alpha^*) = S_\alpha^*$ for all $i \in Q_0$ and $\alpha \in Q_1$.*

Proof. To see the existence of such a homomorphism, it suffices to show that the linear maps P_i, S_α and S_α^* satisfy the defining relations of the Leavitt path algebra.

For (0), we observe that $P_i \circ P_j = \delta_{ij} P_i$.

For (1), we have that for $p = \cdots \alpha_2\alpha_1 \in Q_\infty$,

$$P_{t(\alpha)} S_\alpha(p) = \delta_{t(\alpha),s(\alpha_2)}\, \delta_{\alpha,\alpha_1} \tau_{>1}(p) = \delta_{\alpha,\alpha_1} \tau_{>1}(p) = S_\alpha(p).$$

Here, we use that if $\alpha = \alpha_1$, then $t(\alpha) = s(\alpha_2)$. Similarly, we have

$$S_\alpha \circ P_{s(\alpha)} = S_\alpha.$$

Similar arguments prove the relation (2).

For (3), we have that

$$S_\alpha S_\beta^*(p) = \delta_{\alpha,\beta} \delta_{t(\beta),s(p)} \tau_{>1}(p\beta) = \delta_{\alpha,\beta} \delta_{t(\alpha),s(p)} p = \delta_{\alpha,\beta} P_{t(\alpha)}(p).$$

For (4), we have that

$$\sum_{\{\alpha \in Q_1 \mid s(\alpha)=i\}} S_\alpha^* S_\alpha(p) = \sum_{\{\alpha \in Q_1 \mid s(\alpha)=i\}} S_\alpha^*(\delta_{\alpha,\alpha_1} \tau_{>1}(p)) = \delta_{i,s(p)} p = P_i(p).$$

Then we are done. □

Denote the action of $L_k(Q)$ on $\mathcal{F}$ by "$.$", that is, $a.u = \rho(a)(u)$ for $a \in L_k(Q)$ and $u \in \mathcal{F}$.

Lemma 3.2. *Let p be a left-infinite path in Q and let γ and η be finite paths of length n and m, respectively. Consider p as an element in $\mathcal{F}$. Then the following statements hold.*

(1) $\gamma.p \neq 0$ *if and only if* $\gamma = \tau_{\le n}(p)$. *Indeed,* $\tau_{\le n}(p).p = \tau_{>n}(p)$.

(2) $\eta^*.p \neq 0$ *if and only if* $s(p) = t(\eta)$, *in which case* $\eta.p = p\eta$.

(3) *If* $t(\gamma) = t(\eta)$, *then* $(\eta^*\gamma).p \neq 0$ *if and only if* $\gamma = \tau_{\le n} p$, *in which case one has* $(\eta^*\gamma).p = \tau_{>n}(p)\eta$.

Proof. For an arrow α, we observe that $\alpha.p = p'$ if $p = p'\alpha$ for some left-infinite path p'; otherwise, $\alpha.p = 0$. Then statement (1) follows. For (2), we observe that $\alpha^*.p = p\alpha$ if $s(p) = t(\alpha)$; otherwise, $\alpha^*.p = 0$. The last statement follows from (1) and (2). □

For a nonzero element u in $\mathcal{F}$, its *normal form* means the expression

$$u = \sum_{i=1}^{l} \lambda_i p_i,$$

where each $\lambda_i \in k$ is nonzero and the left-infinite paths p_i are pairwise distinct.

The following result yields the first class of irreducible representations. In particular, the representation $\mathcal{F}$ turns out to be completely reducible.

Theorem 3.3. *Consider the representation $\mathcal{F}$ of $L_k(Q)$. Then the following statements hold.*

(1) *For each $[p] \in \widetilde{Q}_\infty$, the subspace $\mathcal{F}_{[p]} \subseteq \mathcal{F}$ is an irreducible subrepresentation, which satisfies that $\mathrm{End}_{L_k(Q)}(\mathcal{F}_{[p]}) \simeq k$.*

(2) *Two representations $\mathcal{F}_{[p]}$ and $\mathcal{F}_{[q]}$ are isomorphic if and only if $[p] = [q]$.*

Proof. To see that $\mathcal{F}_{[p]} \subseteq \mathcal{F}$ is a subrepresentation, it suffices to notice that for each left-infinite path p we have $p \sim \tau_{>1}(p)$ and $p \sim p\alpha$ for all arrows α with $t(\alpha) = s(p)$.

To prove that the representation $\mathcal{F}_{[p]}$ is irreducible, take a nonzero subrepresentation $U \subseteq \mathcal{F}_{[p]}$, and a nonzero element $u = \sum_{i=1}^{l} \lambda_i p_i$ in U. Here, the expression of u is its normal form. Take n large enough such that all the $\tau_{\leq n}(p_i)$ are pairwise distinct. Then by Lemma 3.2 (1) we have

$$\tau_{\leq n}(p_1).u = \tau_{\leq n}(p_1).(\lambda_1 p_1) = \lambda_1 \tau_{>n}(p_1).$$

This proves that $p_0 = \tau_{>n}(p_1)$ lies in U. We claim that each $p' \in [p]$ lies in U. Then we are done. We observe that $p' \sim p_0$. Assume that $\tau_{>r}(p') = \tau_{>s}(p_0)$. The equalities $\tau_{>s}(p_0) = \tau_{\leq s}(p_0).p_0$ and $p' = (\tau_{\leq r}(p'))^*.\tau_{>r}(p')$ imply that p' lies in U.

Consider a nonzero homomorphism $\phi\colon \mathcal{F}_{[p]} \to \mathcal{F}_{[q]}$. Since $\mathcal{F}_{[p]}$ is irreducible, ϕ is injective. Let $p' \in [p]$ and write

$$\phi(p') = \sum_{i=1}^{l} \lambda_i q_i$$

in its normal form. We claim that $l = 1$ and $q_1 = p'$. Otherwise, we may assume that $q_1 \neq p'$. Take n large enough such that all the $\tau_{\leq n}(q_i)$ are pairwise distinct and that $x = \tau_{\leq n}(q_1) \neq \tau_{\leq n}(p')$. Then by Lemma 3.2(1) $x.p' = 0$ and $x.\phi(p') = x.(\lambda_1 q_1) = \lambda_1 \tau_{>n}(q_1) \neq 0$. A contradiction!

The above claim proves (2). Moreover, we have shown that a nonzero endomorphism $\phi\colon \mathcal{F}_{[p]} \to \mathcal{F}_{[p]}$ necessarily satisfies that $\phi(p') = \lambda_{p'}\, p'$ with $\lambda_{p'} \in k$ for all $p' \in [p]$. It remains to see that all the $\lambda_{p'}$ are the same, and then we have $\mathrm{End}_{L_k(Q)}(\mathcal{F}_{[p]}) \simeq k$. Take p' and p'' in $[p]$. We assume that $\tau_{>r}(p') = \tau_{>s}(p'')$. We deduce from the equalities

$$\tau_{>s}(p'') = \tau_{\leq s}(p'').p'' \quad \text{and} \quad p' = (\tau_{\leq r}(p'))^*.\tau_{>r}(p')$$

that $\lambda_{p'} = \lambda_{p''}$. □

Example 3.4. Let $n \geq 1$ and let $Q = R_n$ be the quiver consisting of one vertex and n loops. Then the Leavitt path algebra $L(n) = L_k(R_n)$ is the *Leavitt algebra* of order n ([19, 20]).

Consider the case $n = 1$. The algebra $L(1)$ is isomorphic to the Laurent polynomial algebra $k[x, x^{-1}]$. Here, the set Q_∞ consists of a single element, and then the representation $\mathcal{F}$ is irreducible. In fact, $\mathcal{F}$ is one-dimensional, on which x acts as the identity.

Consider the case $n \geq 2$. Then the set $\widetilde{Q}_\infty$ of tail-equivalence classes is uncountable. So we obtain a uncountable family of irreducible representations $\mathcal{F}_{[p]}$ for the Leavitt algebra $L(n)$.

3.2 The representation $\mathcal{N}$

Let k be a field and let Q be a quiver. Denote by Q_0^s the set consisting of all sinks in Q. Denote by $\mathcal{N}$ the vector space over k with a basis given by all the finite paths in Q that terminate at a sink. For each sink i, denote by $\mathcal{N}_i$ the subspace of $\mathcal{N}$ spanned by paths p with $t(p) = i$. Then we have

$$\mathcal{N} = \bigoplus_{i \in Q_0^s} \mathcal{N}_i.$$

We will define a representation of $L_k(Q)$ on $\mathcal{N}$. The construction is similar to the one in the previous subsection.

For each vertex $i \in Q_0$, define a linear map $P_i \colon \mathcal{N} \to \mathcal{N}$ such that

$$P_i(p) = \delta_{i,s(p)} p$$

for finite paths p terminating at some sink.

For each arrow $\alpha \in Q_1$, define a linear map $S_\alpha \colon \mathcal{N} \to \mathcal{N}$ as follows:

$$S_\alpha(p) = 0 \ \text{ if } l(p) = 0, \quad \text{and} \quad S_\alpha(p) = \delta_{\alpha,\alpha_1} \alpha_n \cdots \alpha_2$$

for $p = \alpha_n \cdots \alpha_2 \alpha_1$. We define another linear map $S_\alpha^* \colon \mathcal{N} \to \mathcal{N}$ such that

$$S_\alpha^*(p) = \delta_{t(\alpha),s(p)} p\alpha = \delta_{t(\alpha),s(\alpha_1)} \ p\alpha.$$

Proposition 3.5. *There is an algebra homomorphism $\psi \colon L_k(Q) \to \mathrm{End}_k(\mathcal{N})$ such that $\psi(e_i) = P_i$, $\psi(\alpha) = S_\alpha$ and $\psi(\alpha^*) = S_\alpha^*$ for all $i \in Q_0$ and $\alpha \in Q_1$.*

Proof. The proof is similar to the proof of Proposition 3.1. We note that in verifying relation (4), we use that $P_i(e_j) = 0$ for i regular and $j \in Q_0^s$. □

The following lemma is similar to Lemma 3.2.

Lemma 3.6. *Let p be a finite path in Q that terminates at a sink, and let γ and η be finite paths of length n and m respectively. Consider p as an element in $\mathcal{N}$.*

Then the following statements hold.

(1) $\gamma.p \neq 0$ *if and only if* $\gamma = \tau_{\leq n}(p)$. *Indeed,* $\tau_{\leq n}(p).p = \tau_{>n}(p)$.

(2) $\eta^*.p \neq 0$ *if and only if* $s(p) = t(\eta)$, *in which case* $\eta.p = p\eta$.

(3) *If* $t(\gamma) = t(\eta)$, *then* $(\eta^*\gamma).p \neq 0$ *if and only if* $\gamma = \tau_{\leq n} p$, *in which case one has* $(\eta^*\gamma).p = \tau_{>n}(p)\eta$. □

The following result gives us the second class of irreducible representations of the Leavitt path algebra. In particular, the representation $\mathcal{N}$ turns out to be completely reducible.

Theorem 3.7. *Consider the representation* $\mathcal{N}$ *of* $L_k(Q)$. *Then the following statements hold.*

(1) *For each* $i \in Q_0^s$, *the subspace* $\mathcal{N}_i \subseteq \mathcal{N}$ *is an irreducible subrepresentation, which satisfies that* $\mathrm{End}_{L_k(Q)}(\mathcal{N}_i) \simeq k$.

(2) *Two representations* $\mathcal{N}_i$ *and* $\mathcal{N}_j$ *are isomorphic if and only if* $i = j$.

(3) *For any* $[p] \in \widetilde{Q}_\infty$ *and* $i \in Q_0^s$, $\mathcal{F}_{[p]}$ *is not isomorphic to* $\mathcal{N}_i$.

Proof. The subspace $\mathcal{N}_i \subseteq \mathcal{N}$ is clearly a subrepresentation, and it is generated by the trivial path e_i.

For the irreducibility of $\mathcal{N}_i$, take a nonzero subrepresentation $U \subseteq \mathcal{N}_i$ and a nonzero element

$$u = \sum_{j=1}^{l} \lambda_j p_j \in U$$

in its normal form. That is, each $\lambda_j \in k$ is nonzero, the p_j are pairwise distinct, and $t(p_j) = i$ for all j. We choose the normal form such that p_1 is longest among all the p_j (such p_1 need not be unique). Then by Lemma 3.6 (1) we have $p_1.u = \lambda_1 e_i$. Therefore $e_i \in U$, from which we infer $U = \mathcal{N}_i$. Here, we use "." to denote the action of $L_k(Q)$ on $\mathcal{N}$.

Take a nonzero homomorphism $\phi\colon \mathcal{N}_i \to \mathcal{N}_j$, which is necessarily injective. Write $\phi(e_i) = \sum_{r=1}^{l} \lambda_r p_r$ in its normal form. We claim that $l = 1$ and $p_1 = e_i$. This will imply $i = j$ and $\mathrm{End}_{L_k(Q)}(\mathcal{N}_i) \simeq k$. To prove the claim, we assume the converse. Then we may assume that p_1 is longest among all the p_r. In particular, $l(p_1) \geq 1$. Then by Lemma 3.6 (1) $p_1.e_i = 0$ and $p_1.\phi(e_i) = \lambda_1 e_j \neq 0$. A contradiction!

For (3), it suffices to show that each homomorphism $\phi\colon \mathcal{N}_i \to \mathcal{F}_{[p]}$ satisfies $\phi(e_i) = 0$, whence $\phi = 0$. Otherwise, write the nonzero element $\phi(e_i)$ in its normal form: $\phi(e_i) = \sum_{j=1}^{l} \lambda_j p_j$. Here, all the p_j lie in $[p]$. Take n large enough such that all the truncations $\tau_{\leq n}(p_j)$ are pairwise distinct. Then $\tau_{\leq n}(p_1).e_i = 0$ and by Lemma 3.2 (1) $\tau_{\leq n}(p_1).\phi(e_i) = \lambda_1 \tau_{>n}(p_1) \neq 0$. This is absurd. □

Remark 3.8. We will show that the irreducible representations $\mathcal{N}_i$ are isomorphic to certain minimal left ideals of the Leavitt path algebra; see Proposition 4.3 (2).

4 Minimal left ideals, a faithfulness result and point modules

In this section, we draw some consequences from the constructed representations $\mathcal{F}$ and $\mathcal{N}$. We show that for a countable quiver, the constructed irreducible representations contain all minimal left ideals of the Leavitt path algebra. We prove that for a certain quiver, the representation $\mathcal{F} \oplus \mathcal{N}$ is faithful. We relate the irreducible representations $\mathcal{F}_{[p]}$ to point modules.

4.1 Some consequences

The following result extends slightly a result contained in the proof of [24, Theorem 5.4]. Recall that $L_k(Q) = \bigoplus_{n\in\mathbb{Z}} L_k(Q)_n$ is naturally $\mathbb{Z}$-graded such that the natural algebra homomorphism $\iota: kQ \to L_k(Q)$ is graded. We point out that the injectivity of ι is known; see [18, Lemma 1.6].

Proposition 4.1. *Let Q be an arbitrary quiver. Fix $m, n \geq 0$. Then the following subset of $L_k(Q)_{n-m}$,*

$$\{p^*q \mid p, q \text{ are paths in } Q \text{ with } t(p) = t(q),\ l(p) = m \text{ and } l(q) = n\}, \quad (4.1)$$

is linearly independent. In particular, the algebra homomorphism $\iota: kQ \to L_k(Q)$ is injective.

Proof. The second statement is an immediate consequence of the first one, once we notice that the homomorphism ι preserves the gradings, and that $\{q \mid l(q) = n\}$ is a basis of kQ_n. Here, we use that $\iota(q) = e^*_{t(q)}q$.

Suppose (p_i, q_i), $1 \leq i \leq l$, are pairwise distinct pairs of paths such that each $p_i^*q_i$ is in the set (4.1). Consider an element $u = \sum_{i=1}^{l} \lambda_i p_i^* q_i$ in $L_k(Q)$ with each $\lambda_i \in k$ nonzero. We will show that u is nonzero. Consider the terminating vertex $t(q_1)$ of q_1. Then we are in two cases. In the first case, there is a path p with $s(p) = t(q_1)$ and $t(p)$ a sink. Consider the element pq_1 in $\mathcal{N}_{t(p)}$. By Lemma 3.6 we have that

$$u.(pq_1) = \sum_{\{i \mid 1 \leq i \leq l,\, q_i = q_1\}} \lambda_i pp_i.$$

Observe that the paths pp_i in the summation are pairwise distinct. Then we have $u.(pq_1) \neq 0$, which implies that $u \neq 0$. In the second case, there is a left-infinite path p with $s(p) = t(q_1)$. Consider the element pq_1 in $\mathcal{F}_{[p]}$. Then the same argument as in the first case will work. □

The following observation in the finite case is implicitly contained in [3, Section 3]. Recall that for a quiver Q, Q_0^s denotes the set of all sinks in Q.

Proposition 4.2. *Let Q be an arbitrary quiver. Then the subset*

$$\{p^*q \mid p, q \text{ are finite paths in } Q \text{ with } t(p) = t(q) \in Q_0^s\} \subset L_k(Q) \qquad (4.2)$$

is linearly independent.

Proof. It suffices to show that each element $u = \sum_{i=1}^{l} \lambda_i p_i^* q_i$ in $L_k(Q)$ is nonzero, where each $\lambda_i \in k$ is nonzero and the pairs (p_i, q_i) are pairwise distinct with each $p_i^* q_i$ in the set (4.2). Assume that q_1 is the shortest among the paths q_i (such q_1 need not be unique). Consider the element $q_1 \in \mathcal{N}$. Then by Lemma 3.6 we have

$$u.q_1 = \lambda_1 p_1 + \sum \lambda_i p_i,$$

where the summation runs over $2 \le i \le l$ with $q_i = q_1$. Observe that these p_i are different from p_1. It follows that $u.q_1 \neq 0$. This proves that u is nonzero. □

4.2 Minimal left ideals

We show that some of the irreducible representations constructed in Section 3 are isomorphic to minimal left ideals of the Leavitt path algebra. For this, we recall some terminology from [9, 10]. Let Q be a quiver. A vertex i is called *linear* if there is at most one arrow starting at i and there are no oriented cycles going through i. A linear vertex i is a *line point* if $t(p)$ is a linear vertex for every path p that starts at i.

There are two cases for a line point. A line point i is *infinite* if there is a left-infinite path p starting at i; this unique path is called the *tail* of i. A line point i is *finite* if there is a path from i to a sink; the unique sink is called the *end* of i. We remark that a sink is a finite line point, whose end is itself.

For a vertex i of Q, we consider the left ideal $L_k(Q)e_i$ generated by the idempotent e_i. This left ideal is viewed as a representation of $L_k(Q)$; it is nonzero by Proposition 4.1.

Proposition 4.3. *Let Q be a quiver. Then the following statements hold.*

(1) *Let i be an infinite line point with tail p. Then there is an isomorphism of representations*

$$L_k(Q)e_i \simeq \mathcal{F}_{[p]}.$$

(2) *Let i be a finite line point with end i_0. Then there is an isomorphism of representations*

$$L_k(Q)e_i \simeq \mathcal{N}_{i_0}.$$

We consider a *countable* quiver Q, that is, both the sets of vertices and arrows are countable. By [10, Theorem 4.13], up to isomorphism, all minimal left ideals of $L_k(Q)$ are of the form $L_k(Q)e_i$ for some line point i. Therefore, the irreducible representations constructed in Section 3 contain all minimal left ideals of $L_k(Q)$. It seems that a similar result holds for an arbitrary quiver; see [4, Proposition 1.9 and Theorem 1.10].

Proof. (1) For each left-infinite path q in $[p]$, take $n(q) \geq 0$ smallest such that $\tau_{>n(q)}(q) = \tau_{>m}(p)$ for some $m \geq 0$; such an $m = m(q)$ is unique, since the tail of an infinite line point is not cyclic. We observe that for each pair (n,m) such that $\tau_{>n}(q) = \tau_{>m}(p)$, we have $(\tau_{\leq n}(q))^* \tau_{\leq m}(p) = (\tau_{\leq n(q)}(q))^* \tau_{\leq m(q)}(p)$ in $L_k(Q)$; here, we use relation (4) in Section 2.2 and the fact that each vertex appearing in p is linear.

Define a linear map $\mathcal{F}_{[p]} \to L_k(Q)e_i$, sending q to $(\tau_{\leq n(q)}(q))^* \tau_{\leq m(q)}(p)$. It is a homomorphism of representations by direct verification. Since the homomorphism sends p to e_i, by the irreducibility of $\mathcal{F}_{[p]}$ we deduce that it is an isomorphism.

(2) Let q be the unique path from i to its end i_0. Then we have an isomorphism $L_k(Q)e_{i_0} \to L_k(Q)e_i$ sending x to xq; compare [9, Lemma 2.2]. The inverse is given by the multiplication of q^* from the right. Here, we apply relations (3) and (4) in Section 2.2 to have $qq^* = e_{i_0}$ and $q^*q = e_i$.

Define a linear map $\mathcal{N}_{i_0} \to L_k(Q)e_{i_0}$ sending p to p^*. It sends e_{i_0} to $e_{i_0} = e_{i_0}^*$. The map is a homomorphism of representations by direct verification. Then it follows from the irreducibility of $\mathcal{N}_{i_0}$ that the map is an isomorphism. □

4.3 A faithfulness result

Recall that a quiver Q is row-finite, provided that there is no infinite emitter in Q. A left-infinite path p which is not cyclic is said to be *non-cyclic*. This is equivalent to the condition that $p \neq \tau_{>n}(p)$ for any $n \geq 1$.

We point out that a part of the argument in the following proof resembles the one given in the first step in the proof of [17, Theorem 4.2].

Proposition 4.4. *Let Q be a row-finite quiver. Assume that for each vertex i in Q, there exists either a path which starts at i and terminates at a sink, or a non-cyclic left-infinite path which starts at i. Then the representation $\mathcal{F} \oplus \mathcal{N}$ is faithful.*

Proof. We will show that for each nonzero element $u \in L_k(Q)$, its action on $\mathcal{F} \oplus \mathcal{N}$ is nonzero. Write

$$u = \sum_{i=1}^{l} \lambda_i p_i^* q_i$$

in its normal form; see (2.1). Moreover, there exists $j \in Q_0$ such that $e_j u \neq 0$; see Section 2.2. Observe that if the action of $e_j u$ on $\mathcal{F} \oplus \mathcal{N}$ is nonzero, so does u. So we replace u by $e_j u$. This amounts to the requirement that in the normal form of u, $s(p_i) = j$ for all i.

We use induction on the numerical invariant $\kappa(u)$ introduced in Section 2.2. For the case $\kappa(u) = 0$, we have that $u = \sum_{i=1}^{l} \lambda_i q_i$ and $t(q_i) = j$. Without loss of generality, we assume that q_1 is shortest among all the q_i. Consider the vertex j. Then we are in two cases. In the first case, there is a path p with $s(p) = j$ and $t(p)$ a sink. Then $(pu).(pq_1) = \lambda_1 e_j \neq 0$. Here, we view $pq_1 \in \mathcal{N}$. This shows that pu acts nontrivially on $\mathcal{N}$, and so does u.

In the second case, there is a non-cyclic left-infinite path p with $s(p) = j$. We view $pq_1 \in \mathcal{F}$. Then

$$u.(pq_1) = \sum_{i=1}^{l} \lambda_i q_i.(pq_1).$$

Observe that for $i \neq 1$ we have $q_i.(pq_1) \neq 0$ if and only if $q_i = \tau_{\leq n_i}(p)q_1$ with $n_i = l(q_i) - l(q_1)$, in which case $q_i.(pq_1) = \tau_{>n_i}(p)$ and $n_i \geq 1$. Consequently, by Lemma 3.2 we have

$$u.(pq_1) = \lambda_1 p + \sum \lambda_i \tau_{>n_i}(p),$$

where the summation runs over all $i \neq 1$ such that $q_i = \tau_{\leq n_i}(p)q_1$. Since the left-infinite path p is non-cyclic, in particular, $p \neq \tau_{>m}(p)$ for any $m \geq 1$, we have $u.(pq_1) \neq 0$. This implies that u acts nontrivially on $\mathcal{F}$.

For the general case, we assume that $\kappa(u) > 0$. This implies that $j = s(p_i)$ is not a sink. By assumption, the vertex j is not an infinite emitter, and then the vertex j is regular. By relation (4) in Section 2.2, we have

$$u = e_j u = \sum_{\{\alpha \in Q_1 \mid s(\alpha)=j\}} \alpha^* \alpha u.$$

In particular, there is an arrow α with $v = \alpha u \neq 0$. Observe by relation (3) that $\kappa(v) < \kappa(u)$. Hence by the induction hypothesis, the action of v on $\mathcal{F} \oplus \mathcal{N}$ is nonzero. This forces that the action of u is also nonzero. □

Remark 4.5. (1) The conditions on the quiver are necessary for the proposition. Consider the quiver $Q = R_1$ in Example 3.4, that is, it consists of a vertex with one loop. The representation $\mathcal{F}$ is one dimensional, and $\mathcal{N}$ is zero. The representation $\mathcal{F} \oplus \mathcal{N}$ is not faithful.

(2) One may apply [15, Theorem 8] to simplify the argument above. Indeed, [15, Theorem 8] states that every two-sided ideal of $L_k(Q)$ is generated by elements of

the form $u = e_i + \sum_{j=1}^{l} \lambda_j c_j$, where i is a vertex in Q and each c_j is an oriented cycle passing through i, $\lambda_j \in k$. Hence to prove the proposition above, it suffices to show that such u acts nontrivially on $\mathcal{F} \oplus \mathcal{N}$.

We apply Proposition 4.4 to a finite quiver Q without oriented cycles to recover [3, Proposition 3.5]. For a sink i of Q, denote by n_i the number of paths that terminate at i. We denote by $M_n(k)$ the full $n \times n$ matrix algebra over k.

Proposition 4.6. *Let Q be a finite quiver without oriented cycles. Then there is an isomorphism of algebras*

$$L_k(Q) \simeq \prod_{i \in Q_0^s} M_{n_i}(k).$$

Proof. The Leavitt path algebra $L_k(Q)$ is finite dimensional by Corollary 2.2, and the corresponding representation $\mathcal{F}$ is zero. By Proposition 4.4 the finite dimensional representation $\mathcal{N}$ is faithful and completely reducible. It follows that the Leavitt path algebra $L_k(Q)$ is semi-simple and $\{\mathcal{N}_i \mid i \in Q_0 \text{ is a sink}\}$ is a complete set of pairwise non-isomorphic irreducible representations of $L_k(Q)$, each of which has its endomorphism algebra isomorphic to k; see Theorem 3.7. Observe that $\dim_k \mathcal{N}_i = n_i$. Then the above isomorphism is a direct consequence of the Wedderburn–Artin Theorem for semisimple algebras. □

Remark 4.7. The above isomorphism can be proved directly by combining Lemma 2.1 and Proposition 4.2. Indeed, the Leavitt path algebra $L_k(Q)$ has a basis $\{p^*q \mid p, q \text{ are finite paths in } Q \text{ with } t(p) = t(q) \in Q_0^s\}$.

4.4 Point modules

Let $p = \cdots \alpha_2\alpha_1$ be a left-infinite path in Q. We will relate the irreducible representation $\mathcal{F}_{[p]}$ in Section 3.1 to point modules in [24, 25].

Recall that the path algebra kQ is graded by the length of paths. We define a graded kQ-module M_p associated to p as follows. As a graded vector space, $M_p = \bigoplus_{n \geq 0} kz_n$ with a basis $\{z_n \mid n \geq 0\}$ such that $\deg z_n = n$. The kQ-action is defined such that for each vertex i, $e_i.z_n = z_n$ if $s(\alpha_{n+1}) = i$, and $e_i.z_n = 0$ otherwise; for each arrow α we have $\alpha.z_n = z_{n+1}$ if $\alpha = \alpha_{n+1}$, and $\alpha.z_n = 0$ otherwise. This graded kQ-module M_p is known as the *point module* associated to p; see [24, 25].

Recall that the Leavitt path algebra $L_k(Q)$ is $\mathbb{Z}$-graded, and the natural algebra homomorphism $\iota: kQ \to L_k(Q)$ preserves the grading. Then $L_k(Q) \otimes_{kQ} M_p$ becomes a graded $L_k(Q)$-module. We are interested in this module.

A left-infinite path $p = \cdots \alpha_2\alpha_1$ is *regular* if each vertex $s(\alpha_i)$ is regular.

Lemma 4.8. *The module $L_k(Q) \otimes_{kQ} M_p$ is linearly spanned by the set*

$$S_p = \{\gamma^* \otimes z_m \mid m \geq 0,\ \gamma \text{ finite paths with } t(\gamma) = t(\alpha_m)\},$$

where we identify α_0 with $e_{s(p)}$. If p is regular, then $L_k(Q) \otimes_{kQ} M_p$ is linearly spanned by

$$S'_p = \{\gamma^* \otimes z_m \in S_p \mid \gamma \text{ does not end with } \alpha_m \text{ if } m \geq 1\}.$$

Proof. Observe that the kQ-module M_p is generated by z_0. By Corollary 2.2, $L_k(Q) \otimes_{kQ} M_p$ is spanned by elements of the form $\gamma^*\eta \otimes z_0 = \gamma^* \otimes \eta.z_0$ with $t(\gamma) = t(\eta)$. Observe that in M_p, $\eta.z_0 \neq 0$ if and only if $\eta = \tau_{\leq m}(p)$, where m is the length of η; indeed, $\tau_{\leq m}(p).z_0 = z_m$. This proves the first statement.

Suppose p is regular. We will show that each element in S_p lies in S'_p. Consider $\gamma^* \otimes z_m$ in S_p such that $m \geq 1$ and $\gamma = \alpha_m \gamma'$ for some path γ'. Since the vertex $s(\alpha_m)$ is regular, by relation (4) in Section 2.2

$$\begin{aligned} e_{s(\alpha_m)} \otimes z_{m-1} &= \sum_{\{\alpha \in Q_1 \mid s(\alpha) = s(\alpha_m)\}} \alpha^*\alpha \otimes z_{m-1} \\ &= \sum_{\{\alpha \in Q_1 \mid s(\alpha) = s(\alpha_m)\}} \alpha^* \otimes \alpha.z_{m-1} \\ &= \alpha_m^* \otimes z_m, \end{aligned}$$

where the last equality uses the fact that $\alpha.z_{m-1} = 0$ for each arrow $\alpha \neq \alpha_m$ and $\alpha_m.z_{m-1} = z_m$. Then we have

$$\gamma^* \otimes z_m = \gamma'^* \alpha_m^* \otimes z_{m-1} = \gamma'^* e_{s(\alpha_m)} \otimes z_{m-1} = \gamma'^* \otimes z_{m-1}.$$

By induction on m, we infer that $\gamma^* \otimes z_m$ lies in S'_p. □

Consider the set

$$\{(n, q) \mid n \in \mathbb{Z},\ q \in Q_\infty \text{ such that } \tau_{>m-n}(q) = \tau_{>m}(p) \text{ for some } m\}. \quad (4.3)$$

Let $\mathcal{F}_p$ be the vector space spanned by this set. Then $\mathcal{F}_p$ is naturally graded by means of $\deg(n, q) = n$.

We endow $\mathcal{F}_p$ with a graded $L_k(Q)$-module structure: for each vertex i and arrow α in Q we have that $e_i.(n, q) = (n, P_i(q))$, $\alpha.(n, q) = (n+1, S_\alpha(q))$ and $\alpha^*.(n, q) = (n-1, S_\alpha^*(q))$. Here, the operators P_i, S_α and S_α^* are defined in Section 3.1, and we identify $(n, 0)$ with the zero element in $\mathcal{F}_p$. Similar to Proposition 3.1, this defines a $L_k(Q)$-module structure on $\mathcal{F}_p$.

The following result relates the irreducible representation $\mathcal{F}_{[p]}$ to the graded module $\mathcal{F}_p$, and then to the point module M_p. In particular, if p is regular and irrational, we have an isomorphism $L_k(Q) \otimes_{kQ} M_p \simeq \mathcal{F}_{[p]}$ of $L_k(Q)$-modules.

Proposition 4.9. *Keep the notation as above. Then we have the following statements.*

(1) *There is a surjective homomorphism of graded $L_k(Q)$-modules*

$$L_k(Q) \otimes_{kQ} M_p \to \mathcal{F}_p;$$

it is an isomorphism if p is regular.

(2) *There is a surjective homomorphism $\pi_p\colon \mathcal{F}_p \to \mathcal{F}_{[p]}$ of $L_k(Q)$-modules sending (n, q) to q; π_p is an isomorphism if and only if p is irrational.*

Proof. For (1), observe a graded kQ-module homomorphism $M_p \to \mathcal{F}_p$ by sending z_m to $(m, \tau_{>m}(p))$. This homomorphism extends to a graded $L_k(Q)$-module homomorphism $\phi_p\colon L_k(Q) \otimes_{kQ} M_p \to \mathcal{F}_p$.

Consider the element $\gamma^* \otimes z_m$ in S_p with n the length of γ. We have

$$\phi_p(\gamma^* \otimes z_m) = \gamma^*.(m, \tau_{>m}(p)) = (m - n, \tau_{>m}(p)\gamma).$$

For each (n, q) in the set (4.3), take m to be the minimal nonnegative integer such that $q = \tau_{>m}(p)\gamma$ for a path γ with length $m - n$. By the identity above, we have $\phi_p(\gamma^* \otimes z_m) = (n, q)$, proving that ϕ_p is surjective.

Assume that p is regular. We define a linear map $\psi_p\colon \mathcal{F}_p \to L_k(Q) \otimes_{kQ} M_p$ by $\psi_p(n, q) = \gamma^* \otimes z_m$. Then the composite $\psi_p \circ \phi_p$ is the identity on the set S'_p. By Lemma 4.8, $\psi_p \circ \phi_p$ is the identity map. Hence, the homomorphism ϕ_p is injective, which proves (1).

Statement (2) is obvious. For the last statement, it suffices to recall the following fact: a left-infinite path p is irrational if and only if for each left-infinite path q in its tail-equivalence class $[p]$, there is a unique integer n such that

$$\tau_{>m-n}(q) = \tau_{>m}(p)$$

for some m. □

Remark 4.10. If p is regular, then the isomorphisms ϕ_p and ψ_p imply that S'_p in Lemma 4.8 is a linear basis of $L_k(Q) \otimes_{kQ} M_p$.

5 Algebraic branching systems

In this section, we relate the irreducible representations constructed in Section 3 to certain algebraic branching systems in [17]. This somehow is expected by the authors; see the second paragraph in [17, p. 259]. For a row-finite quiver, we classify algebraic branching systems whose associated representations of the Leavitt path algebra are irreducible. It turns out that all these irreducible representations are isomorphic to the ones in Section 3.

Let Q be an arbitrary quiver. Following [17, Definition 2.1], a *Q-algebraic branching system* consists of a set X, and a family of its subsets

$$\{X_i, X_\alpha \mid i \in Q_0, \alpha \in Q_1\}$$

together with a bijection $\sigma_\alpha \colon X_{t(\alpha)} \to X_\alpha$ for each arrow α, where the subsets are subject to the following constraints:

(1) $X_i \cap X_j = \varnothing = X_\alpha \cap X_\beta$ for $i \neq j, \alpha \neq \beta$,

(2) $X_\alpha \subseteq X_{s(\alpha)}$ for each $\alpha \in Q_1$,

(3) $X_i = \bigcup_{\{\alpha \in Q_1 \mid s(\alpha)=i\}} X_\alpha$ for each regular vertex i.

We will denote the above Q-algebraic branching system simply by X. We point out that this notion is closely related to dynamical systems with partitions studied in [13].

A Q-algebraic branching system X is *saturated* provided that $X = \bigcup_{i \in Q_0} X_i$; it is said to be *perfect*, if, in addition, (3) also holds when i is an infinite emitter. For a row-finite quiver Q, every saturated Q-algebraic branching system is perfect.

Let X and Y be two Q-algebraic branching systems. A map $f \colon X \to Y$ is a *morphism* of Q-algebraic branching systems if $f(X_i) \subseteq Y_i$ and $f(X_\alpha) \subseteq Y_\alpha$ for all vertices i and arrows α of Q, and f is compatible with the bijections inside X and Y. Two Q-algebraic branching systems are *isomorphic* provided that there exist mutually inverse morphisms between them.

Examples of Q-algebraic branching systems are given in [17, Theorem 3.1]. We are interested in the following examples, both of which are perfect.

Example 5.1. (1) Let p be a left-infinite path in Q. Consider its tail-equivalence class $[p]$ as a set. It is a Q-algebraic branching system in the following manner: $[p]_i = \{q \in [p] \mid s(q) = i\}$ and $[p]_\alpha = \{q \in [p] \mid q$ starts with $\alpha\}$. The bijection $\sigma_\alpha \colon [p]_{t(\alpha)} \to [p]_\alpha$ sends q to $q\alpha$.

(2) Let $i \in Q_0^s$ be a sink. Consider the set N_i consisting of paths in Q that terminate at i. It is a Q-algebraic branching system in a similar manner.

We recall that one may associate a representation of the Leavitt path algebra to each Q-algebraic branching system. Let X be a Q-algebraic branching system. Denote by $\mathcal{M}(X)$ the vector space consisting of all functions from X to k, which vanish on all but finitely many elements in X. For each $x \in X$, denote by $\chi_x \colon X \to k$ the *characteristic function*. That is, $\chi_x(y) = \delta_{x,y}$ for all x and y in X. Then $\{\chi_x \mid x \in X\}$ is a basis of $\mathcal{M}(X)$.

The module $\mathcal{M}(X)$ in the following lemma differs from the module in [17, Theorem 2.2], but is the same as the one mentioned in [17, Remark 2.3]. Here, we adapt the notation for our convenience.

Lemma 5.2. *Let X be a Q-algebraic branching system. Then there is a representation of $L_k(Q)$ on $\mathcal{M}(X)$ as follows:*

(1) *for each $i \in Q_0$, $e_i.\chi_x = \chi_x$ if $x \in X_i$, otherwise $e_i.\chi_x = 0$,*

(2) *for each $\alpha \in Q_1$, $\alpha.\chi_x = \chi_{\sigma_\alpha^{-1}(x)}$ if $x \in X_\alpha$, otherwise $\alpha.\chi_x = 0$,*

(3) *for each $\alpha \in Q_1$, $\alpha^*.\chi_x = \chi_{\sigma_\alpha(x)}$ if $x \in X_{t(\alpha)}$, otherwise $\alpha^*.\chi_x = 0$.*

For a Q-algebraic branching system X, the above representation $\mathcal{M}(X)$ of $L_k(Q)$ is said to be the *associated representation* to X. Observe that X is saturated if and only if the associated representation $\mathcal{M}(X)$ is unital, that is,

$$L_k(Q).\mathcal{M}(X) = \mathcal{M}(X).$$

Let $f\colon X \to Y$ be a morphism of Q-algebraic branching systems. Assume that X is perfect. Then f induces a homomorphism of associated representations

$$\mathcal{M}(f)\colon \mathcal{M}(X) \longrightarrow \mathcal{M}(Y),$$

which sends χ_x to $\chi_{f(x)}$. Here, we use the facts that

$$f^{-1}(Y_i) = X_i \quad \text{and} \quad f^{-1}(Y_\alpha) = X_\alpha$$

for each vertex i and arrow α of Q, which is derived directly from the perfectness of X. The homomorphism $\mathcal{M}(f)$ is an isomorphism if and only if so is f.

The following observation shows that the representations constructed in Section 3 are associated to the Q-algebraic branching systems in Example 5.1.

Proposition 5.3. *Let Q be a quiver. Use the notation as above. Then there are isomorphisms of representations*

$$\mathcal{F}_{[p]} \simeq \mathcal{M}([p]) \quad \textit{and} \quad \mathcal{N}_i \simeq \mathcal{M}(N_i)$$

for each left-infinite path p and sink i.

Proof. The linear map $\mathcal{F}_{[p]} \to \mathcal{M}([p])$ sending q to χ_q is an isomorphism of representations. This is done by direct verification. The same map works for $\mathcal{N}_i$. □

We infer from Section 3 and Proposition 5.3 that the representations associated to algebraic branching systems in Example 5.1 are irreducible. In some cases, these are all the irreducible representations constructed in this way.

Theorem 5.4. *Let Q be a quiver and X be a perfect Q-algebraic branching system. Then the associated representation $\mathcal{M}(X)$ is irreducible if and only if X is isomorphic to $[p]$ or N_i, where p is a left-infinite path and i is a sink in Q.*

This result implies that for a row-finite quiver Q, all the irreducible representations associated to some saturated Q-algebraic branching systems are isomorphic to the ones constructed in Section 3.

The following example shows that the perfectness condition in the above theorem is necessary.

Example 5.5. Let Q be the following quiver consisting of two vertices $\{1, 2\}$ and infinitely many arrows from 1 to 2:

$$1 \xrightarrow{\infty} 2.$$

Consider the Q-algebraic branching system $X = \{*\}$ consisting of a single element, such that $X_1 = X$, $X_2 = \varnothing = X_\alpha$ for each arrow α. Then X is saturated but not perfect; thus it is isomorphic to none of the Q-algebraic branching systems in Example 5.1. However, the associated representation $\mathcal{M}(X)$ is one-dimensional and therefore irreducible. We refer to [2, Lemma 1.2] for the structure of the Leavitt path algebra $L_k(Q)$.

We make some preparation for the proof of Theorem 5.4. The argument here resembles the one in the proof of [13, Theorem 1]. Let X be a perfect Q-algebraic branching system, and let $x \in X$. If $x \in X_i$ for a non-sink i, then there exists a unique arrow α such that $s(\alpha) = i$ and $x \in X_\alpha$; thus there exists a unique $y \in X_{t(\alpha)}$ such that $\sigma_\alpha(y) = x$. We repeat this argument for y. Then we infer that for each element $x \in X$ there are two cases as follows.

In the first case, there exists a unique left-infinite path $p(x) = \cdots \alpha_n \cdots \alpha_2 \alpha_1$, such that there exist $x_m \in X_{s(\alpha_{m+1})}$ for $m \geq 0$, such that

$$x = x_0 \quad \text{and} \quad \sigma_{\alpha_m}(x_m) = x_{m-1} \ \text{ for } m \geq 1.$$

Here, we notice that $X_{s(\alpha_m)} = X_{t(\alpha_{m-1})}$ for $m \geq 1$.

In the second case, there exists a unique path $p(x) = \alpha_l \cdots \alpha_2 \alpha_1$ terminating at a sink such that there exist $x_m \in X_{s(\alpha_{m+1})}$ for $0 \leq m \leq l-1$, and $x_l \in X_{t(\alpha_l)}$, satisfying that $x = x_0$ and $\sigma_{\alpha_m}(x_m) = x_{m-1}$ for $1 \leq m \leq l$. The length l of the path $p(x)$ might be zero; this happens if and only if $x \in X_i$ for a sink i.

Recall that

$$Q_\infty = \bigcup_{[p] \in \widetilde{Q}_\infty} [p]$$

is a disjoint union. Then it is naturally a Q-algebraic branching system as in Example 5.1 (1). Similarly, the disjoint union $N = \bigcup_{i \in Q_0^s} N_i$ is a Q-algebraic branching system, and so is the disjoint union $Q_\infty \cup N$.

We have the following observation, whose proof is routine.

Lemma 5.6. *Let X be a perfect Q-algebraic branching system. Then the map*

$$f_X: X \longrightarrow Q_\infty \cup N, \quad f_X(x) = p(x)$$

is a morphism of Q-algebraic branching systems.

We are in a position to prove Theorem 5.4.

Proof of Theorem 5.4. The "if" part follows from Proposition 5.3 and Section 3. For the "only if" part, assume that the associated representation $\mathcal{M}(X)$ is irreducible. The morphism in Lemma 5.6 induces a nonzero homomorphism

$$\mathcal{M}(f_X): \mathcal{M}(X) \to \mathcal{M}(Q_\infty \cup N);$$

it is injective, since $\mathcal{M}(X)$ is irreducible. Observe from Proposition 5.3 that

$$\mathcal{M}(Q_\infty \cup N) \simeq \mathcal{F} \oplus \mathcal{N}.$$

Recall from Section 3 that the representation $\mathcal{F} \oplus \mathcal{N}$ is completely reducible and each irreducible summand occurs with multiplicity one. It follows that any irreducible subrepresentation of $\mathcal{F} \oplus \mathcal{N}$ equals $\mathcal{F}_{[p]}$ or $\mathcal{N}_i$ for some left-infinite path p or a sink i. From these we infer that the image of the injective homomorphism $\mathcal{M}(f_X)$ equals $\mathcal{F}_{[p]}$ or $\mathcal{N}_i$. This implies that the image of f_X equals $[p]$ or N_i, and then as Q-algebraic branching systems, X is isomorphic to $[p]$ or N_i. □

6 Twisted representations

In this section we study representations $\mathcal{F}_{[p]}^{\mathbf{a}}$ and $\mathcal{N}_i^{\mathbf{a}}$ of $L_k(Q)$ that are obtained by twisting the irreducible representations in Section 3 with automorphisms that scale the actions of the arrows. In particular, we obtain new irreducible representations for rational tail-equivalence classes. In the end, we prove the faithfulness of some completely reducible representation.

Let Q be an arbitrary quiver. Denote by $k^\times$ the *multiplicative group* of k, and by $(k^\times)^{Q_1}$ the product group. Its elements are of the form $\mathbf{a} = (a_\alpha)_{\alpha \in Q_1}$ with each $a_\alpha \in k^\times$, and its multiplication is componentwise. For each $\mathbf{a}$, there is an algebra automorphism $\gamma_{\mathbf{a}}: L_k(Q) \to L_k(Q)$ such that $\gamma_{\mathbf{a}}(e_i) = e_i$, $\gamma_{\mathbf{a}}(\alpha) = a_\alpha \alpha$, and $\gamma_{\mathbf{a}}(\alpha^*) = a_\alpha^{-1} \alpha^*$. This gives rise to an injective group homomorphism

$$\gamma: (k^\times)^{Q_1} \to \mathrm{Aut}(L_k(Q)).$$

This is called the *(generalized) scaling action*; compare [12, Definition 2.13].

For an element $\mathbf{a} = (a_\alpha)_{\alpha \in Q_1}$ and a nontrivial path $p = \alpha_n \cdots \alpha_2 \alpha_1$ in Q, set $a_p = a_{\alpha_n} \cdots a_{\alpha_2} a_{\alpha_1}$. The element $\mathbf{a}$ is called *p-stable* if $a_p = 1$.

Recall that for a representation M of an algebra A and an automorphism σ of A, we have the *twisted representation* M^σ as follows: $M^\sigma = M$ as vector spaces, and the action is given by $a.m^\sigma = (\sigma(a).m)^\sigma$. Here, for an element m in M, we denote by m^σ the corresponding element in M^σ. Moreover, the representation M^σ is irreducible if and only if M is.

For the Leavitt path algebra, we write the twisted representation $M^{\gamma_{\mathbf{a}}}$ simply as $M^{\mathbf{a}}$. Observe that $M^{\mathbf{1}} = M$.

Recall the irreducible representations $\mathcal{F}_{[p]}$ and $\mathcal{N}_i$ constructed in Section 3. We are interested in their twisted representations $\mathcal{F}^{\mathbf{a}}_{[p]}$ and $\mathcal{N}^{\mathbf{a}}_i$.

Proposition 6.1. *Let Q be a quiver, and let $\mathbf{a}, \mathbf{b} \in (k^\times)^{Q_1}$. We use the notation as above. Then the following statements hold.*

(1) *For $[p] \in \widetilde{Q}_\infty$ an irrational class, the representations $\mathcal{F}^{\mathbf{a}}_{[p]}$ and $\mathcal{F}^{\mathbf{b}}_{[p]}$ are isomorphic.*

(2) *For $[q^\infty] \in \widetilde{Q}_\infty$ a rational class with q a simple oriented cycle, the representations $\mathcal{F}^{\mathbf{a}}_{[q^\infty]}$ and $\mathcal{F}^{\mathbf{b}}_{[q^\infty]}$ are isomorphic if and only if $\mathbf{ab}^{-1}$ is q-stable.*

(3) *For $i \in Q_0^s$ a sink, the representations $\mathcal{N}^{\mathbf{a}}_i$ and $\mathcal{N}^{\mathbf{b}}_i$ are isomorphic.*

Proof. To show (1), it suffices to prove that

$$\mathcal{F}_{[p]} \simeq \mathcal{F}^{\mathbf{a}}_{[p]}$$

for every $\mathbf{a} \in (k^\times)^{Q_1}$. Fix $p_0 \in [p]$. Then for each $q \in [p]$, we may choose natural numbers n and m such that $\tau_{>n}(q) = \tau_{>m}(p_0)$. Since the left-infinite path p_0 is irrational, the number $n - m$ is unique for q. For the same reason, the scalar $\theta(q) := (a_{\tau_{\leq n}(q)})^{-1} a_{\tau_{\leq m}(p_0)}$ is independent of the choice of n and m. Then we have the required isomorphism $\phi\colon \mathcal{F}_{[p]} \to \mathcal{F}^{\mathbf{a}}_{[p]}$, which sends $q \in [p]$ to $\theta(q)q$. One proves (3) with a similar argument.

To see (2), it suffices to prove that

$$\mathcal{F}_{[q^\infty]} \simeq \mathcal{F}^{\mathbf{a}}_{[q^\infty]}$$

if and only if $\mathbf{a}$ is q-stable. For the "only if" part, we observe that every isomorphism

$$\phi\colon \mathcal{F}_{[q^\infty]} \to \mathcal{F}^{\mathbf{a}}_{[q^\infty]}$$

satisfies $\phi(q^\infty) = \lambda q^\infty$ for some nonzero scalar λ; consult the third paragraph in the proof of Theorem 3.3. Then

$$\phi(q^\infty) = \phi(q.q^\infty) = q.\phi(q^\infty) = \lambda a_q q^\infty.$$

This implies that $a_q = 1$.

Finally, we consider the "if" part. For each $p \in [q^\infty]$, take the smallest natural number n_0 such that $\tau_{>n_0}(p) = q^\infty$, and set $\theta(p) = (a_{\tau_{\leq n_0}(p)})^{-1}$; in addition, set $\theta(q^\infty) = 1$. Define a linear map $\phi\colon \mathcal{F}_{[q^\infty]} \to \mathcal{F}^{\mathbf{a}}_{[q^\infty]}$ sending p to $\theta(p)p$. It is routine to verify that this is an isomorphism of representations. Here, one needs for the verification to use that $\mathbf{a}$ is q-stable. □

To summarize, we list all the irreducible representations of the Leavitt path algebra that are constructed in this paper. To this end, we fix for each rational class $[p] \in \widetilde{Q}^{\mathrm{rat}}_\infty$ a simple oriented cycle $q = \alpha_n \cdots \alpha_2\alpha_1$ with $p \sim q^\infty$. For each $\lambda \in k^\times$, set $\mathbf{a}_{\lambda,q} = (a_\alpha)_{\alpha\in Q_1}$ such that $a_{\alpha_1} = \lambda$ and $a_\alpha = 1$ for $\alpha \neq \alpha_1$.

Set $\mathcal{F}^{\lambda}_{[p]} = \mathcal{F}^{\mathbf{a}_{\lambda,q}}_{[p]}$. By Proposition 6.1 (2) we have that for each $\mathbf{a} \in (k^\times)^{Q_1}$,

$$\mathcal{F}^{\mathbf{a}}_{[p]} \simeq \mathcal{F}^{a_q}_{[p]};$$

moreover, $\mathcal{F}^{\lambda}_{[p]}$ is isomorphic to $\mathcal{F}^{\lambda'}_{[p]}$ if and only if $\lambda = \lambda'$. Observe that we have $\mathcal{F}^{1}_{[p]} = \mathcal{F}_{[p]}$.

We obtain a list of pairwise non-isomorphic irreducible representations for the Leavitt path algebra $L_k(Q)$. The representations are parameterized by the disjoint union $\widetilde{Q}^{\mathrm{irr}}_\infty \cup (k^\times \times \widetilde{Q}^{\mathrm{rat}}_\infty) \cup Q^s_0$.

Theorem 6.2. *Let Q be a quiver and let k be field. Then the following set,*

$$\{\mathcal{F}_{[p]} \mid [p] \in \widetilde{Q}^{\mathrm{irr}}_\infty\} \cup \{\mathcal{F}^{\lambda}_{[p]} \mid \lambda \in k^\times,\ [p] \in \widetilde{Q}^{\mathrm{rat}}_\infty\} \cup \{\mathcal{N}_i \mid i \in Q^s_0\},$$

consists of pairwise non-isomorphic irreducible representations of $L_k(Q)$.

Proof. It suffices to show that these representations are pairwise non-isomorphic. This follows from Theorem 3.3 (2), Theorem 3.7 (3) and Proposition 6.1 (2). Here, we need to use the same argument in Theorem 3.3 to show that

$$\mathcal{F}^{\lambda}_{[p]} \simeq \mathcal{F}^{\lambda'}_{[p']} \implies [p] = [p'].$$

Moreover, $\mathcal{F}^{\lambda}_{[p]}$ is neither isomorphic to $\mathcal{F}_{[p']}$ with $[p']$ irrational, nor isomorphic to $\mathcal{N}_i$ with i a sink. We omit the details. □

We close this paper with a faithfulness result on the following completely reducible representation:

$$\mathcal{S} = \bigoplus_{[p]\in\widetilde{Q}^{\mathrm{irr}}_\infty} \mathcal{F}_{[p]} \bigoplus_{\lambda\in k^\times,\,[p]\in\widetilde{Q}^{\mathrm{rat}}_\infty} \mathcal{F}^{\lambda}_{[p]} \bigoplus_{i\in Q^s_0} \mathcal{N}_i .$$

This partially remedies the counterexample in Remark 4.5.

Proposition 6.3. *Let Q be a row-finite quiver, and let k be an infinite field. Then the representation $\mathcal{S}$ of $L_k(Q)$ is faithful.*

Proof. We observe that a modified argument in the proof of Proposition 4.4 will work. It suffices to show that any nonzero element $u = \sum_{i=1}^{l} \lambda_i q_i$ in $L_k(Q)$ acts nontrivially on $\mathcal{S}$. Here, u is in its normal form (see (2.1)), and $\kappa(u) = 0$, that is, all the q_i are paths in Q. We may assume that $t(q_i) = j$ for some $j \in Q_0$ and all $1 \leq i \leq l$. Without loss of generality, we assume that q_1 is shortest among all the q_i.

By Proposition 4.4 and its proof, we may assume that there is a cyclic path $p = q^\infty$ starting at j with q a simple oriented cycle.

Consider pq_1 as an element in $\mathcal{F}_{[p]}^{\lambda}$ for some λ. Consider

$$I_1 = \{i \mid 2 \leq i \leq l,\, q_i = q^{m_i} q_1 \text{ for some } m_i \geq 1\},$$
$$I_2 = \{2, 3, \ldots, l\} \setminus I_1.$$

Here, $l(q)m_i = l(q_i) - l(q_1)$ for $i \in I_1$. Then by a variant of Lemma 3.2, we have

$$\begin{aligned} u.(pq_1) &= \lambda_1 p + \sum_{i \in I_1} \lambda_i\, q_i.(pq_1) + \sum_{i \in I_2} \lambda_i\, q_i.(qq_1) \\ &= \Big(\lambda_1 + \sum_{i \in I_1} \lambda_i \lambda^{m_i}\Big) p + \sum_{i \in I_2} \lambda_i\, q_i.(qq_1). \end{aligned}$$

We observe that in the summation indexed by I_2, $q_i.(qq_1)$ is either zero or a multiple of a path in $\mathcal{F}_{[p]}^{\lambda}$ that is different from p. Since the field k is infinite, we may take $\lambda \in k^\times$ such that

$$\lambda_1 + \sum_{i \in I_1} \lambda_i \lambda^{m_i} \neq 0.$$

In this case, we have that in $\mathcal{F}_{[p]}^{\lambda}$, $u.(pq_1) \neq 0$. We are done. □

Remark 6.4. For a finite field k, the representation $\mathcal{S}$ might not be faithful. Such an example is given by $Q = R_1$ in Example 3.4, the quiver consisting of one vertex with one loop.

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