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Dual Gabriel theorem with applications

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Abstract We introduce the quiver of a bicomodule over a cosemisimple coalgebra. Applying this to the coradical C_0 of an arbitrary coalgebra C, we give an alternative definition of the Gabriel quiver of C, and then show that it coincides with the known Ext quiver of C and the link quiver of C. The dual Gabriel theorem for a coalgebra with a separable coradical is obtained, which generalizes the corresponding result for a pointed coalgebra. We also give a new description of $C_1 = C_0 \wedge_C C_0$ of any coalgebra C, which can be regarded as a generalization of the first part of the well-known Taft-Wilson Theorem for pointed coalgebras. As applications, we give a characterization of locally finite coalgebras via their Gabriel quivers, and a property of the Gabriel quiver of a quasi-coFrobenius coalgebra.

Keywords: quivers, cotensor coalgebra, quasi-coFrobenius coalgebra.

1 Introduction and preliminaries

1.1 Quivers and algebras

In the representation theory of finite-dimensional algebras, the quiver is a fundamental tool.

A finite-dimensional algebra A over a field K is called elementary if the quotient algebra of A modulo the Jacobson radical is isomorphic to a product of K as K-algebras, and called basic if this quotient is isomorphic to a product of division K-algebras. A theorem due to Gabriel says that an elementary K-algebra A is isomorphic to the factor algebra of the path algebra $KQ(A)^a$ by an admissible ideal, where $Q(A)$ is the Gabriel quiver of A (see Theorem 1.9, in ref. [1] and p. 43 of ref. [2]). Since any finite-dimensional algebra is Morita equivalent to a uniquely determined basic algebra, and a basic algebra over an algebraically closed field is elementary, it follows that any finite-dimensional algebra A over an algebraically closed field is Morita equivalent to $KQ(A)^a$ modulo an admissible ideal. On the other hand, the Auslander-Reiten quiver of a finite-dimensional algebra A, which is defined by the indecomposable A-modules and irreducible maps, is an essential approach and technique in studying the representations of A (see, e.g. refs. [1, 2]).

1.2 Quivers and coalgebras

As pointed out by Chin and Montgomery in ref. [3], by the fundamental theorem for coalgebras (i.e. every comodule is a sum of its finite-dimensional subcomodules; in particular, every simple coalgebra is finite-dimensional), it is reasonable to expect that the quiver technique for algebras could be extended to coalgebras.

In fact, in the past few years, there were several works towards this direction. The path algebra construction has been dualized by Chin and Montgomery to get a path coalgebra; the Ext quiver of coalgebra C has been introduced and then a dual version of the Gabriel theorem for coalgebras has been given in ref. [3] (here C is not necessarily finite-dimensional). Montgomery also introduces the link quiver of coalgebra C by using the wedge of simple subcoalgebras of C. This link quiver is isomorphic to the Ext quiver, up to multiple arrows, and it is connected if and only if C is an indecomposable coalgebra; using this she proved that a pointed Hopf algebra is a crossed product of a group algebra over the indecomposable component of the identity element, see ref. [4]. On the other hand, the almost split sequences and the Auslander-Reiten quivers for coalgebras turn out to be also very useful in studying the comodules of coalgebras. See refs. [5, 6].

There are also several works to construct neither commutative nor cocommutative Hopf algebras via quivers. In ref. [7], Cibils determined all the graded Hopf structures with the length grading on a path algebra KZ_n^a of a basic cycle Z_n. In ref. [8], Cibils and Rosso studied the graded Hopf structures on path algebras. In ref. [9] Green and Solberg studied the Hopf structures on some special quadratic quotients of path algebras. More recently, Cibils and Rosso[10] introduced the notion of the Hopf quivers and then classified all the graded Hopf algebras with the length grading on path coalgebras. Using the quiver technique all the monomial Hopf algebras have been classified in ref. [11], and in ref. [12] a class of bi-Frobenius algebras which are not Hopf algebras has been constructed via quivers.

1.3 Main results in this paper

These quoted works inspire us to pay more attention to the quiver method towards coalgebras. Note that in the algebra case the Gabriel quiver has an alternative definition rather than the extensions of simple modules. In this paper, we first introduce the quiver of a bicomodule over a cosemisimple coalgebra. Applying this to the C_0-C_0-bicomodule C_1/C_0, we give an alternative definition of the Gabriel quiver $Q(C)$ of C, where C is an arbitrary coalgebra and $C_1 = C_0 \wedge_C C_0$ (see, e.g. ref. [13]); and then show that $Q(C)$ coincides with the Ext quiver of C introduced by Chin and Montgomery[3]. This will be done in sec. 2.

By definition a coalgebra C is called pointed if each simple subcoalgebra of C is of dimension one (in the finite-dimensional case, this is exactly the dual of an elementary algebra), and called basic if the dual of each simple subcoalgebra of C is a finite-dimensional division K-algebra. As a dual of the result due to Gabriel as quoted in 1.1, Chin and Montgomery proved that any pointed coalgebra is isomorphic to a large subcoalgebra of the path coalgebra of the Ext quiver of C (for the notion of "large subcoalgebra" see Remark 3.1 in this paper). Since any coalgebra is Morita-Takeuchi equivalent to a uniquely determined basic coalgebra, and a basic coalgebra over an algebraically closed field is pointed, it follows that any coalgebra C over an algebraically closed field is Morita-Takeuchi equivalent to a large subcoalgebra of the path coalgebra of the Ext quiver of C. See Theorem 4.3 in ref. [3]. In Section 3 (Theorem 3.1), we prove that a coalgebra C (over an arbitrary field K) with a separable coradical C_0 is isomorphic to a large subcoalgebra of the cotensor coalgebra $\mathrm{Cot}_{C_0}(C_1/C_0)$. Note that C_0 is always separable over an algebraically closed field, and if C is pointed then $\mathrm{Cot}_{C_0}(C_1/C_0)$ is isomorphic to the path coalgebra kQ^c of the Gabriel quiver Q of C. In this way the dual of the Gabriel theorem for pointed coalgebras is extended to the one for the coalgebras with separable coradicals.

For an arbitrary coalgebra C with the coradical $C_0 = \oplus_{i\in I} D^i$, where D^i's are simple subcoalgebras of C, we show in Section 4 that there hold

$$C_1 = \sum_{i,j\in I} (D^i \wedge_C D^j)$$

and

$$C_1/C_0 \cong \oplus_{i,j\in I} (D^i \wedge_C D^j)/(D^i + D^j).$$

See Theorem 4.1. This can be regarded as a generalization of the first part of the well-known Taft-Wilson Theorem for pointed coalgebras, see Remark 4.1. As an application we unify the link quiver of a coalgebra with the Gabriel quiver and the Ext quiver (Corollary 4.1).

In the last two sections, we include two applications of Theorem 3.1 and Theorem 4.1, by claiming that a coalgebra with a separable coradical is locally finite if and only if its Gabriel quiver is locally finite (Theorem 5.1); and the Gabriel quiver of a non-simple quasi-coFrobenius coalgebra has no sources and no sinks (Theorem 6.1). In the finite-dimensional case, Theorem 6.1 is dual to the corresponding one for algebras.

In the following, all coalgebras and all tensor products are over a fixed field

K. For a K-space V, denote the dual $\mathrm{Hom}_K(V, K)$ by V^*.

1.4 Cotensor coalgebras

Let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra over the ground field K, where Δ_C and ε_C are the structure maps. A right C-comodule (M, ρ) is a vector space M endowed with a structure map $\rho : M \longrightarrow M \otimes C$ such that $(\rho \otimes Id) \circ \rho = (I \otimes \Delta_C) \circ \rho$ and $(Id \otimes \varepsilon_C) \circ \rho = Id$, where Id denotes the identity map. Similarly one has left C-comodules. Let D be a coalgebra. By a D-C-bicomodule (M, ρ_l, ρ_r) we mean that (M, ρ_l) is a left D-comodule and (M, ρ_r) is a right C-comodule, satisfying $(Id \otimes \rho_r) \circ \rho_l = (\rho_l \otimes Id) \circ \rho_r$.

Let (M, ρ) and (N, δ) be a right and a left C-comodules, respectively. Then the cotensor product of M and N over C is defined to be the subspace of $M \otimes N$ given by

$$M \square_C N = \mathrm{Ker}(\rho \otimes Id - Id \otimes \delta : M \otimes N \longrightarrow M \otimes C \otimes N)$$

If M is a D-C-bicomodule and N is a C-D'-bicomodule, then $M \square_C N$ is a D-D'-bicomodule. The cotensor product is associative, i.e., if in addition L is a D'-C'-bicomodule, then $(M \square_C N) \square_{D'} L \simeq M \square_C (N \square_{D'} L)$ as D-C'-bicomodules.

Let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra, and (M, ρ_l, ρ_r) a C-C-bicomodule. Write $\rho_l(m) = \sum m_{-1} \otimes m_0$ and $\rho_r(m) = \sum m_0 \otimes m_1$ for every $m \in M$. Define $M^{\square 0} = C$, $M^{\square 1} = M$ and $M^{\square n} = (M^{\square n-1}) \square_C M$ for any $n \geqslant 2$. Note that $M^{\square n}$ is a subspace of $M^{\otimes n}$ for all $n \geqslant 1$. If $\sum m^1 \otimes \cdots \otimes m^n \in M^{\square n}$, we write it as $\sum m^1 \square \cdots \square m^n$. Define the cotensor coalgebra $\mathrm{Cot}_C(M)$. As a vector space, $\mathrm{Cot}_C(M) = \oplus_{i=0}^{\infty} M^{\square i}$. The counit ε is given by $\varepsilon|_{M^{\square i}} = 0$ for $i \geqslant 1$ and $\varepsilon|_{M^{\square 0}} = \varepsilon_C$; the comultiplication Δ of $\mathrm{Cot}_C(M)$ is defined as $\Delta|_{M^{\square 0}} = \Delta_C$, $\Delta(m) = \rho_l(m) + \rho_r(m) = \sum m_{-1} \otimes m_0 + m_0 \otimes m_1$ for all $m \in M$, and in general, if $\sum m^1 \square \cdots \square m^n \in M^{\square n}$ $(n \geqslant 1)$, then

$$\begin{aligned}\Delta(\sum m^1 \square \cdots \square m^n) = & \sum (m^1)_{-1} \otimes ((m^1)_0 \square \cdots \square m^n) \\ & + \sum_{i=1}^{n-1} (m^1 \square \cdots \square m^i) \otimes (m^{i+1} \square \cdots \square m^n) \\ & + \sum (m^1 \square \cdots \square (m^n)_0) \otimes (m^n)_1 \\ & \in (C \otimes M^{\square n}) \oplus \bigoplus_{i=1}^{n-1} (M^{\square i} \otimes M^{\square (n-i)}) \oplus (M^{\square n} \otimes C) \\ & \subseteq \mathrm{Cot}_C M \otimes \mathrm{Cot}_C(M).\end{aligned}$$

One can verify that Δ is well-defined and $(\mathrm{Cot}_C(M), \Delta, \varepsilon)$ is a coalgebra.

Remark 1.1. In the case that C is cosemisimple, the coalgebra $\mathrm{Cot}_C(M)$ is coradically graded, i.e., $\{\oplus_{i \leqslant n} M^{\square i} | n = 0, 1, \cdots\}$ is its coradical filtration (see sec. 2 of ref. [14]).

1.5 Path coalgebras

By a quiver, we mean an oriented graph $Q = (Q_0, Q_1, s, t)$ with Q_0 being the set of vertices and Q_1 being the set of arrows, where s, t are two maps from Q_1 to Q_0. For $\alpha \in Q_1$, $s(\alpha)$ and $t(\alpha)$ denote the starting and terminating vertices of α, respectively. Note that the quivers considered here could be infinite.

Recall that the path coalgebra KQ^c of a qiver Q is defined as follows (see ref. [3]). As a vector space KQ^c has a basis consisting of paths in Q, the comultiplication is given by

$$\Delta(p) = \alpha_l \cdots \alpha_1 \otimes s(\alpha_1) + \sum_{i=1}^{l-1} \alpha_l \cdots \alpha_{i+1} \otimes \alpha_i \cdots \alpha_1 + t(\alpha_l) \otimes \alpha_l \cdots \alpha_1$$

for each path $p = \alpha_l \cdots \alpha_1$ with each $\alpha_i \in Q_1$ and $\varepsilon(p) = 0$ if $l \geqslant 1$ and 1 if $l = 0$. Then KQ^c is a pointed coalgebra with the coradical filtration $C_n = KQ_0 \oplus \cdots \oplus KQ_n$, where KQ_n is the K-space with the set of all paths of length n.

Remark 1.2. Note that a path coalgebra is a special case of a cotensor coalgebra: for every quiver Q, $KQ^c \simeq \mathrm{Cot}_{kQ_0}(KQ_1)$ as coalgebras, where the bicomodule structure of KQ_1 is given by $\rho_l(\alpha) := t(\alpha) \otimes \alpha$ and $\rho_r(\alpha) := \alpha \otimes s(\alpha)$ for each $\alpha \in Q_1$.

On the other hand, if C is a pointed coalgebra, then C_1/C_0 becomes a C_0-C_0-bicomodule and $KQ(C)^c \simeq \mathrm{Cot}_{C_0}(C_1/C_0)$, where $Q(C)$ is the Gabriel quiver of C, as defined in 2.2 below, see also Remark 3.1 below.

2 The Gabriel quiver of a coalgebra

2.1 Quiver of a bicomodule over a cosemisimple coalgebra

Let D be a cosemisimple K-coalgebra and let (M, ρ_l, ρ_r) be a D-D-bicomodule. We associate a quiver with the bicomodule M.

Write $D = \oplus_{i \in I} D^i$, where each D^i is a simple subcoalgebra of D. Set

$${}^iM^j = \{\, m \in M \mid \rho_l(m) \in D^i \otimes M, \quad \rho_r(m) \in M \otimes D^j \}$$

for each $i, j \in I$. Then $M = \oplus_{i,j \in I}\, {}^iM^j$, and each ${}^iM^j$ is naturally a D^i-D^j-bicomodule, and hence a $(D^j)^*$-$(D^i)^*$-bimodule. Since each $(D^i)^*$ is a simple algebra, it follows that $(D^i)^* \simeq M_{n_i}(\Delta_i)$, where Δ_i is a finite-dimensional division algebra over K. For each i, e_i is fixed to be an idempotent of $(D^i)^*$. Set

$$t_{ij} = \dim_K(e_i.{}^jM^i.e_j)$$

for each pair of $i, j \in I$, where the dots denote the module action. Note that t_{ij} is independent of the choice of e_i's because they are mutually conjugate in $M_{n_i}(\Delta_i)$.

Define the quiver $Q(D, M)$ of the D-D-bicomodule M as follows: the set of vertices is I, and for any $i, j \in I$, the number of arrows from i to j is t_{ij}.

Remark 2.1. (1) We admit the case that I is an infinite set and t_{ij} is infinite, i.e. the quiver $Q(D, M)$ could be infinite.

(2) If D is group-like (i.e. the set of group-like elements $G = G(D)$ forms a basis for D, or equivalently, D is cosemisimple pointed), then the quiver $Q(D, M)$ is simply interpreted as follows. The set of vertices is G, and for any $g, h \in G$, the number of arrows from g to h is t_{gh}, where $t_{gh} = \dim_K {}^hM^g$, and ${}^hM^g = \{m \in M \mid \rho_l(m) = h \otimes m, \quad \rho_r(m) = m \otimes g\}$.

2.2 The Gabriel quiver of a coalgebra

Let (C, Δ) be a coalgebra with a coradical filtration $\{C_n\}$. Set $\pi_0 : C \longrightarrow C/C_0$ to be the canonical projection. Define the map $\tilde{\rho}_l : C \longrightarrow C \otimes C/C_0$ by $\tilde{\rho}_l = (Id \otimes \pi_0) \circ \Delta$, and the map $\tilde{\rho}_r : C \longrightarrow C/C_0 \otimes C$ by $\tilde{\rho}_r = (\pi_0 \otimes Id) \circ \Delta$. Since $\tilde{\rho}_l(C_0) = 0$, $\tilde{\rho}_r(C_0) = 0$, and $\Delta(C_1) \subseteq C_0 \otimes C_1 + C_1 \otimes C_0$, it follows that $\tilde{\rho}_l$ and $\tilde{\rho}_r$ induce two maps $\rho_l : C_1/C_0 \longrightarrow C_0 \otimes C_1/C_0$, and $\rho_r : C_1/C_0 \longrightarrow C_1/C_0 \otimes C_0$, respectively. It is clear that $(C_1/C_0, \rho_l, \rho_r)$ is a C_0-C_0-bicomodule.

Definition 2.1. The Gabriel quiver $Q(C)$ of a coalgebra C is defined to be the quiver $Q(C_0, C_1/C_0)$ of C_0-C_0-bicomodule C_1/C_0.

More precisely, let $C_0 = \oplus_{i \in I} D^i$, where D^i's are simple subcoalgebras, and e_i be a fixed primitive idempotent of $(D^i)^*$. Then the vertices of $Q(C_0, C_1/C_0)$ are $i \in I$, and there are $t_{ij} = \dim_K e_i.{}^j(C_1/C_0)^i.e_j$ arrows from i to j.

2.3 The Ext quiver of a coalgebra

Let C be a K-coalgebra. Recall the definition of the Ext quiver of C introduced by Chin and Montgomery[3]. Let $\{S_i \mid i \in I\}$ be a complete set of isoclasses of the right simple C-comodules. The Ext quiver of C is an oriented graph with vertices indexed by I, and there are $\dim_K \mathrm{Ext}^1(S_i, S_j)$ arrows from i to j for any $i,\ j \in I$.

Note that on p. 468 of ref. [6], the Ext quiver is also called the Gabriel quiver. In ref. [4] Montgomery also introduced the link quiver of coalgebra C by using the wedge of simple subcoalgebras of C. This link quiver is isomorphic to the Ext quiver, up to multiple arrows, see Theorem 1.7 in ref. [4]. Moreover, it is shown by Corollary 2.2 in ref. [4] that the Ext quiver of C is connected if and only if C is an indecomposable coalgebra.

The main result of this section is

Theorem 2.1. The Gabriel quiver of C coincides with the Ext quiver of C.

2.4 Some lemmas

In order to prove the result, we need some preparations.

For a right C-comodule M, denote by $E(M)$ its injective hull, which always exists and $\mathrm{soc}(M) = \mathrm{soc}(E(M))$ (see ref. [15] or Chap. 2 of ref. [16]). Let $\{S_i \mid i \in I\}$ be a complete set of isoclasses of the right simple C-comodules.

Then as a right C-comodule we have $D^i \simeq n_i S_i$ and $C \simeq \oplus_{i\in I} E(D^i) \simeq \oplus_{i\in I} n_i E(S_i)$. Note that $(D^i)^* \simeq M_{n_i}(\Delta_i)$, where Δ_i is a finite-dimensional division algebra over K.

Lemma 2.1. Suppose $\dim_K \Delta_i = d_i$ for each $i \in I$. We have

$$soc(E(D^i)/D^i) = \oplus_{j\in I} \frac{n_i t_{ji}}{d_j} S_j$$

and

$$soc(E(S_i)/S_i) \simeq \oplus_{j\in I} \frac{t_{ji}}{d_j} S_j.$$

Proof. Recall that $(C_1/C_0, \rho_l, \rho_r)$ is a C_0-C_0-bicomodule. Set

$$^i(C_1/C_0) = \{x \in C_1/C_0 \mid \rho_l(x) \in D^i \otimes (C_1/C_0)\}.$$

Then $^i(C_1/C_0)$ is a left D^i-comodule.

We may identify C with $\oplus_{i\in I} E(D^i)$. It follows that as a right C-comodule we have

$$C_1 = \oplus_{i\in I}(E(D^i) \cap C_1).$$

(In fact, for each $c \in C_1$, we have $c = \sum c_i$ with $c_i \in E(D^i)$. Since $\Delta(c_i) \in E(D^i) \otimes C$, it follows that $\Delta(c_i) = \sum_j d_{ij} \otimes c_{ij}$ with $d_{ij} \in E(D^i)$ and $\{d_{ij}\}$ is linearly independent for each $i \in I$. Then $\Delta(c) = \sum_{i,j} d_{ij} \otimes c_{ij} \in C_1 \otimes C_1$. Since $\{d_{ij}\}$ is linearly independent, it follows that each c_{ij} is contained in C_1, and hence by the counitary property, each $c_i \in C_1$.)

Thus

$$C_1/C_0 = \oplus_{i\in I}(E(D^i) \cap C_1)/D^i$$

as right C_0-comodules, and hence $(E(D^i) \cap C_1)/D^i$ is a cosemisimple C_0-comodule, which implies $(E(D^i) \cap C_1)/D^i \subseteq \mathrm{soc}(E(D^i)/D^i)$. While $\mathrm{soc}(C/C_0) = C_1/C_0$ (see p. 64 of ref. [17]), it follows that

$$\begin{aligned}\mathrm{soc}(C/C_0) &= \mathrm{soc}((\oplus_{i\in I} E(D^i))/(\oplus_{i\in I} D^i)) = \oplus_{i\in I}\mathrm{soc}(E(D^i)/D^i)\\ &= C_1/C_0 = \oplus_{i\in I}(E(D^i) \cap C_1)/D^i.\end{aligned}$$

This forces

$$\mathrm{soc}(E(D^i)/D^i) = (E(D^i) \cap C_1)/D^i.$$

We claim $\mathrm{soc}(E(D^i)/D^i) \subseteq {}^i(C_1/C_0)$, and then by $C_1/C_0 = \oplus_{i\in I} {}^i(C_1/C_0)$ we have

$$\mathrm{soc}(E(D^i)/D^i) = {}^i(C_1/C_0).$$

To see this, note that

$$\tilde{\rho}_l((E(D^i)) \subseteq E(D^i) \otimes C, \quad \tilde{\rho}_l(C_1) \subseteq C_0 \otimes C_1,$$

it follows that

$$\tilde{\rho}_l((E(D^i) \cap C_1) \subseteq (E(D^i) \cap C_0) \otimes C_1.$$

Note that $E(D^i) \cap C_0 \subseteq \mathrm{soc}(E(D^i)) = D^i$, it follows that

$$\tilde{\rho}_l(E(D^i) \cap C_1) \subseteq D^i \otimes C_1$$

and hence

$$\rho_l(\mathrm{soc}(E(D^i)/D^i)) = \rho_l((E(D^i)\cap C_1)/D^i) \subseteq D^i \otimes (C_1/C_0).$$

That is $\mathrm{soc}(E(D^i)/D^i) \subseteq {}^i(C_1/C_0)$. This proves the assertion.

Note that ${}^i(C_1/C_0)^j$ is a D^i-D^j-bicomodule, and hence a $(D^j)^*$-$(D^i)^*$-bimodule. Thus ${}^i(C_1/C_0)^j.e_i$ is a left $(D^j)^*$-module, and hence a right D^j-comodule, where e_i is a primitive element of $(D^i)^*$. Thus we have

$${}^i(C_1/C_0)^j.e_i = m_j S_j$$

as a right D^j-comodule, for some non-negative integer m_j. Since $t_{ji} = \dim_K(e_j.{}^i(C_1/C_0)^j.e_i)$ and $\dim_K S_j = n_j d_j$, it follows that

$$m_j n_j d_j = \dim_K {}^i(C_1/C_0)^j.e_i = \dim_K n_j(e_j.{}^i(C_1/C_0)^j.e_i) = n_j t_{ji}$$

and hence $m_j = \frac{t_{ji}}{d_j}$. It follows that

$$\begin{aligned}\mathrm{soc}(E(D^i)/D^i) &= {}^i(C_1/C_0)\\ &= \oplus_{j\in I}{}^i(C_1/C_0)^j\\ &= \oplus_{j\in I} n_i\ {}^i(C_1/C_0)^j e_i\\ &= \oplus_{j\in I}\frac{n_i t_{ji}}{d_j}S_j.\end{aligned}$$ □

2.5 Proof of Theorem 2.1

Since $D^j \simeq n_j S_j$, it suffices to compute $\mathrm{Ext}^1(S_i, D^j)$. For this, we take a minimal injective resolution of D^j (see ref. [18])

$$0 \longrightarrow D^j \xrightarrow{d_0} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} E_2 \longrightarrow \cdots,$$

where $E_0 = E(D^j)$ and $E_1 = E(E_0/D^j)$. Since

$$\mathrm{Im}(d_0) = \mathrm{soc}(E_0), \quad \mathrm{Im}(d_1) \supseteq \mathrm{soc}(E_1),$$

it follows that for every comodule map $g : S_i \longrightarrow E_0$ we have $d_1 \circ g = 0$, and that for every comodule map $f : S_i \longrightarrow E_1$ we have $d_2 \circ f = 0$. It follows that

$$\mathrm{Ext}^1(S_i, D^j) = \mathrm{Hom}_C(S_i, E_1) = \mathrm{Hom}_C(S_i, soc(E_0/D^j)),$$

here we have used the fact $\mathrm{soc}(E_1) = \mathrm{soc}(E_0/D^j)$. By Lemma 2.1 we have $\mathrm{soc}(E_0/D^j) = \oplus_{i\in I}\frac{n_j t_{ij}}{d_i}S_i$. Combining this with the fact $\mathrm{Hom}_C(S_i, S_i) = \Delta_i$, we have

$$t_{ij} = \frac{1}{n_j}\dim_K \mathrm{Ext}^1(S_i, D^j) = \dim_K \mathrm{Ext}^1(S_i, S_j).$$

This completes the proof. □

Remark 2.2. Recall that two coalgebras C and D are said to be Morita-Takeuchi equivalent, if the categories of C-comodules and D-comodules are equivalent (see ref. [19]). Then by Theorem 2.1 two coalgebras have the same Gabriel quiver provided that they are Morita-Takeuchi equivalent.

3 The dual Gabriel theorem

3.1 The main result of this section

Let L be a field extension of K, and C a K-coalgebra. Then $C \otimes L$ is naturally an L-coalgebra. A coalgebra is called separable provided that $C \otimes L$ is cosemisimple for any field extension L. Note that C is separable if and only if $C \otimes C^{cop}$ is cosemisimple. (In fact, the cosemisimple coalgebras are a direct sum of simple coalgebras, thus this follows by dualizing, Theorem 6.1.2 of ref. [20]). For example, a group-like coalgebra C is separable. If K is of characteristic zero, then any cosemisimple coalgebra is separable. Note that the coradical C_0 of C is separable if C is pointed or if K is algebraically closed.

The main result of this section is

Theorem 3.1. Let C be a coalgebra with a separable coradical C_0. Then there exists a coalgebra embedding $i: C \hookrightarrow \mathrm{Cot}_{C_0}(C_1/C_0)$ with $i(C_1) = C_0 \oplus C_1/C_0$.

3.2 Some lemmas

To prove Theorem 3.1, one needs the following fundamental lemma, which gives the universal mapping property of cotensor coalgebras.

Let C and D be coalgebras and $f: D \longrightarrow C$ be a coalgebra map. Then D becomes a C-C-bicomodule via f: the left and right comodule structure maps are given by $(f \otimes Id) \circ \Delta_D$ and $(Id \otimes f) \circ \Delta_D$, respectively.

Lemma 3.1. Let C and D be coalgebras and M a C-C-bicomodule. Given a coalgebra map $f_0 : D \longrightarrow C$, and a C-C- bicomodule map $f_1 : D \longrightarrow M$ with the property that f_1 vanishes on the coradical D_0 of D, where the C-C-bicomodule structure D is given via f_0. Then there exists a unique coalgebra map

$$F : D \longrightarrow \mathrm{Cot}_C(M)$$

with $\pi_i \circ F = f_i$ $(i = 0, 1)$, where each $\pi_i : \mathrm{Cot}_C(M) \longrightarrow M^{\Box i}$ is the canonical projection.

Proof. Set Δ_0 to be the identity map of D, $\Delta_1 = \Delta_D$ and define $\Delta_{n+1} = (\Delta_D \otimes Id) \circ \Delta_n$ for all $n \geqslant 1$, where Id denotes the identity map of $D^{\otimes n}$. It is easy to check that $\Delta_n(D) \subseteq D^{\Box n+1}$ and $f_1^{\otimes n+1} \circ \Delta_n(D) \subseteq M^{\Box n+1}$ for each $n \geqslant 1$. We claim that $F : D \longrightarrow \mathrm{Cot}_C(M)$, given by

$$F(d) = f_0(d) + \sum_{n=0}^{\infty} f_1^{\otimes n+1} \circ \Delta_n(d)$$

for each $d \in D$, is well-defined.

In fact, we have $\bigcup_{n \geq 0} D_n = D$, where $\{D_n\}$ is the coradical filtration of D (see, e.g. Theorem 5.2.2 of ref. [17]). Thus for each $d \in D_n$, $f_1^{\otimes m+1}\Delta_m(d) = 0$

for all $m \geqslant n$. This is because

$$\Delta_m(D_n) \subseteq \sum_{i_0+i_1+\cdots+i_m=n} D_{i_0} \otimes \cdots \otimes D_{i_m}$$

and f_1 vanishes on D_0. Thus F is well-defined. Moreover, F is a coalgebra map with $\pi_i \circ F = f_i$ $(i = 0, 1)$.

It remains to prove the uniqueness of coalgebra map $F : D \longrightarrow \mathrm{Cot}_C(M)$ with $\pi_i \circ F = f_i$ $(i = 0, 1)$. Set $f_n = \pi_n \circ F$ for each n. It suffices to prove that $f_n = f_1^{\otimes n} \circ \Delta_{n-1}$ for every $n \geqslant 1$. Use the induction on n. Assume that $f_m = f_1^{\otimes m} \circ \Delta_{m-1}$, $m \geqslant 1$. Consider f_{m+1}. For every $d \in D$, write $\Delta_D(d) = \sum d_1 \otimes d_2$. Since F is a coalgebra map, it follows that $\Delta(F(d)) = (F \otimes F)\Delta_D(d)$. Writing out the both sides explicitly we have

$$\Delta(F(d)) = \sum_n \Delta(f_n(d))$$

with

$$\Delta(f_n(d)) \in C \otimes M^{\square n} \oplus M \otimes M^{\square(n-1)} \oplus \cdots \oplus M^{\square(n-1)} \otimes M \oplus M^{\square n} \otimes C;$$

and

$$\begin{aligned}(F \otimes F)\Delta_D(d) &= \sum_{(d)} F(d_1) \otimes F(d_2) \\ &= \sum_n \sum_{(d), i+j=n} f_i(d_1) \otimes f_j(d_2)\end{aligned}$$

with

$$\begin{aligned}&\sum_{(d), i+j=n} f_i(d_1) \otimes f_j(d_2) \\ &\in C \otimes M^{\square n} \oplus M \otimes M^{\square(n-1)} \oplus \cdots \oplus M^{\square(n-1)} \otimes M \oplus M^{\square n} \otimes C.\end{aligned}$$

It follows that

$$\Delta(f_n(d)) = \sum_{(d), i+j=n} f_i(d_1) \otimes f_j(d_2), \quad \forall n \geqslant 2.$$

Note that $f_n(d) \in M^{\square n}$, and $f_i(d_1) \otimes f_j(d_2) \in M^{\square i} \otimes M^{\square j}$. By the definition of the comultiplication Δ of $\mathrm{Cot}_C(M)$ and by comparing the terms belonging to $M^{\square i} \otimes M^{\square j}$ with $i \neq 0 \neq j$ and $i + j = n$, we obtain

$$f_n(d) = \sum_{(d)} f_i(d_1) \otimes f_j(d_2).$$

In particular we have by induction

$$\begin{aligned}f_{m+1}(d) &= \sum_{(d)} f_m(d_1) \otimes f_1(d_2) \\ &= \sum f_1^{\otimes m} \circ \Delta_{m-1}(d_1) \otimes f_1(d_2) \\ &= f_1^{\otimes m+1} \circ \Delta_m(d).\end{aligned}$$

This completes the proof. □

To complete the proof of Theorem 3.1, we also need the dual Wedderburn-Malcev theorem (see Theorem 2.3.11 of ref. [21] or Theorem 5.4.2 of ref. [17]) and another lemma due to Heyneman-Radford (see ref. [22] or Theorem 5.3.1 of ref. [17]).

Lemma 3.2 (Dual Wedderburn-Malcev theorem)**.** Let C be a coalgebra with a separable coradical. Then there is a coideal I such that $C = C_0 \oplus I$, i.e. there is a coalgebra projection $\pi : C \longrightarrow C_0$ such that $\pi|_{C_0} = Id$.

Lemma 3.3 (Heyneman-Radford)**.** Let C and D be coalgebras and $f : C \longrightarrow D$ a coalgebra map. Then f is injective if and only if $f|_{C_1}$ is injective.

3.3 Proof of the main result

Now we are ready to prove Theorem 3.1.

By the dual Wedderburn-Malcev theorem, there is a coideal I of C such that $C = C_0 \oplus I$. Thus we have a coalgebra projection $f_0 : C \longrightarrow C_0$ such that $f_0|_{C_0} = Id$. Note that C becomes a C_0-C_0-bicomodule via f_0, I is a C_0-C_0-subbicomodule of C. Set $C_{(1)} = C_1 \cap I$. Then $C_1 = C_0 \oplus C_{(1)}$. Note that $C_{(1)}$ is a C_0-C_0-subbicomodule of I and the canonical vector space isomorphism $\theta : C_{(1)} \simeq C_1/C_0$ is a C_0-C_0-bicomodule map.

View I as a right $C_0 \otimes {C_0}^{cop}$-comodule and $C_{(1)}$ its subcomodule. Since C_0 is separable, it follows that there exists a $C_0 \otimes {C_0}^{cop}$-comodule decomposition $I = C_{(1)} \oplus J$. Thus we have a C_0-C_0-bicomodule projection $p : I \longrightarrow C_{(1)}$ such that $p|_{C_{(1)}} = Id$. Define a map $f_1 = \theta \circ p \circ f_0'$ from C to C_1/C_0 where $f_0' : C \longrightarrow I$ is the canonical projection. Clearly $f_1 : C \longrightarrow C_1/C_0$ is a C_0-C_0-bicomodule map vanishing on C_0. Thus, by Lemma 3.1 we obtain a unique coalgebra map $i : C \longrightarrow \mathrm{Cot}_{C_0}(C_1/C_0)$ such that $\pi_0 \circ i = f_0$ and $\pi_1 \circ i = f_1$. Clearly $i(C_1) = C_0 \oplus C_1/C_0$. By Lemma 3.3, i is injective. This completes the proof. $\square$

Remark 3.1. (1) Note that if C is pointed, then $\mathrm{Cot}_{C_0}(C_1/C_0)$ is isomorphic to the path coalgebra kQ^c of the Gabriel quiver Q of C.

(In order to see this, just note that KQ^c and $\mathrm{Cot}_{C_0}(C_1/C_0)$ both have the universal mapping property, and then the assertion follows from Lemma 3.1.)

It follows from the above result that a pointed coalgebra can be embedded in the path coalgebra of the Gabriel quiver of C. This has been obtained by Chin and Montgomery in Theorem 4.3 of ref. [3]. See also Corollary 1 of ref. [23].

(2) Recall that a subcoalgebra D of a coalgebra C is said to be large provided that D contains C_1. By the definition of the Gabriel quiver, a large subcoalgebra D of C has the same Gabriel quiver as C. Then Theorem 3.1 says that a coalgebra C (not necessarily finite-dimensional) with a separable coradical is isomorphic to a large subcoalgebra of the cotensor coalgebra $\mathrm{Cot}_{C_0}(C_1/C_0)$.

Recall that any finite-dimensional elementary algebra A is isomorphic to the path algebra of the Gabriel quiver of A modulo an admissible ideal (see, e.g. Theorem 1.9 of ref. [1] or p. 43 of ref. [2]). Thus, Theorem 3.1 can be regarded as a generalization of the dual of this basic result for algebras (note the condition "large" in Theorem 3.1 just corresponds to the condition "admissible" in the case for algebras).

4 A description of C_1

4.1 The wedge

Let C be a coalgebra. Following ref. [13], the wedge of two subspaces V and W of C is defined to be the subspace

$$V \wedge_C W := \{c \in C \mid \Delta_C(c) \in V \otimes C + C \otimes W\}.$$

Let C_0 be the coradical of C_0, i.e., C_0 is the sum of of all simple subcoalgebras of C. Recall that by the definition $C_n = C_0 \wedge_C C_{n-1}$ for $n \geqslant 1$, $\{C_n\}$ is called the coradical filtration of C. Then C_n is a subcoalgebra of C with $C_n \subseteq C_{n+1}$, $C = \cup_{n\geqslant 0} C_n$, and $\Delta C_n \subseteq \sum_{0\leqslant i\leqslant n} C_i \otimes C_{n-i}$ (see, e.g. 5.2.2 of ref. [17]). For the properties of wedges, see Chap. 9 in ref. [13] and sec. 2 in ref. [22]).

4.2 The main result in this section

Let (C, Δ, ε) be a coalgebra with a dual algebra C^*. For $c \in C$ and $f \in C^*$, define

$$f \rightharpoonup c = \sum c_1 f(c_2)$$

and

$$c \leftharpoonup f = \sum f(c_1) c_2,$$

where $\Delta(c) = \sum c_1 \otimes c_2$. Then it is well-known that C becomes a C^*-C^*-bimodule with $c = \varepsilon \rightharpoonup c = c \leftharpoonup \varepsilon$ (see, e.g. 1.6.5 of ref. [17]).

The following result gives a new description of C_1. We will use it in the next section, but also it seems to be of independent interest.

Theorem 4.1. Let C be a coalgebra with a coradical $C_0 = \oplus_{i\in I} D^i$, where D^i are simple subcoalgebras of C. Then

(i) $C_1 = \sum_{i,j\in I}(D^i \wedge_C D^j)$;

(ii) $(D^i \wedge_C D^j) \cap C_0 = D^i + D^j, \quad \forall i, j \in I$;

(iii) $C_1/C_0 \cong \oplus_{i,j\in I}(D^i \wedge_C D^j)/(D^i + D^j)$;

(iv) $(D^i \wedge_C D^j)/(D^i + D^j) \cong {}^i(C_1/C_0)^j, \quad \forall i, j \in I$.

Remark 4.1. Recall that the set of group-like elements of a coalgebra C is $G(C) := \{\ 0 \neq c \in C \mid \Delta(c) = c \otimes c\ \}$, and that a coalgebra C is said to be pointed if each simple subcoalgebra of C is of dimension one. Note that C is pointed if and only if $C_0 = KG(C)$. For $g, h \in G(C)$, denote by $P_{g,h}(C) := \{c \in C \mid \Delta(c) = c \otimes g + h \otimes c\}$, the set of g, h-primitive elements in C. A g, h-primitive element c is said to be non-trivial if $c \notin K(g - h)$.

Let $P'_{g,h}(C)$ be a subspace of $P_{g,h}(C)$ such that

$$P_{g,h}(C) = P'_{g,h}(C) \oplus K(g-h).$$

Then the first part of the Taft-Wilson Theorem for the pointed coalgebras says that if C is pointed, then

$$C_1 = KG(C) \oplus (\bigoplus_{g,h} P'_{g,h}(C)),$$

and hence

$$C_1/C_0 = \bigoplus_{g,h} P'_{g,h}(C) = \bigoplus_{g,h} (Kh \wedge_C Kg)/(Kg + Kh).$$

(For the last equality, see, e.g. Lemma 4.2 of ref. [24]) From this point of view, Theorem 4.1 (iii) can be regarded as a form of the first part of the Taft-Wilson Theorem in general case.

4.3 Proof of Theorem 4.1

(i) On the one hand, we have

$$C_1 = C_0 \wedge_C C_0 = \left(\sum_{i\in I} D^i\right) \wedge_C \left(\sum_{j\in I} D^j\right) \supseteq \sum_{i,j\in I} (D^i \wedge_C D^j).$$

On the other hand, by an elementary argument in the linear algebra, we can write $C = C_0 \oplus V$ with a subspace V such that $\varepsilon(V) = 0$. Take $\varepsilon_i \in C^*$ such that

$$\varepsilon_i|_{D^i} = \varepsilon, \quad \varepsilon_i|_{D^j \oplus V} = 0 \quad (j \neq i).$$

Then

$$\varepsilon(c) = \sum_{i\in I} \varepsilon_i(c), \quad \forall\, c \in C,$$

and hence by the counitary property we have

$$c = \sum_{i,j\in I} (\varepsilon_j \rightharpoonup c \leftharpoonup \varepsilon_i), \quad \forall\, c \in C.$$

While for $c \in C_1$ we claim

$$\varepsilon_j \rightharpoonup c \leftharpoonup \varepsilon_i \in D^i \wedge_C D^j,$$

and then the assertion follows.

In order to see the claim, for $c \in C_1$, consider $\Delta^3(c) = (\Delta \otimes Id \otimes Id)(Id \otimes \Delta)\Delta(c)$. For simplicity we omit the summation in the following

$$\begin{aligned}
\Delta(c) &= c_1 \otimes c_2 \in \sum_s C_1 \otimes D^s + \sum_t D^t \otimes C_1;\\
\Delta^2(c) &= (Id \otimes \Delta)\Delta(c) = c_1 \otimes c_{21} \otimes c_{22}\\
&\in \sum_s C_1 \otimes D^s \otimes D^s + \sum_{t,k} D^t \otimes C_1 \otimes D^k + \sum_{t,k} D^t \otimes D^k \otimes C_1;\\
\Delta^3(c) &= (\Delta \otimes Id \otimes Id)(Id \otimes \Delta)\Delta(c) = c_{11} \otimes c_{12} \otimes c_{21} \otimes c_{22}\\
&\in \sum_s C_1 \otimes C_0 \otimes D^s \otimes D^s + \sum_s C_0 \otimes C_1 \otimes D^s \otimes D^s\\
&\quad + \sum_{t,k} D^t \otimes D^t \otimes C_1 \otimes D^k + \sum_{t,k} D^t \otimes D^t \otimes D^k \otimes C_1.
\end{aligned}$$

By definition we have

$$\Delta(\varepsilon_j \rightharpoonup c \leftharpoonup \varepsilon_i) = \varepsilon_i(c_{11})\varepsilon_j(c_{22})c_{12} \otimes c_{21}.$$

If

$$c_{11} \otimes c_{12} \otimes c_{21} \otimes c_{22} \in \sum_s C_1 \otimes C_0 \otimes D^s \otimes D^s,$$

then

$$\varepsilon_i(c_{11})\varepsilon_j(c_{22})c_{12} \otimes c_{21} \in C_0 \otimes D^j;$$

if

$$c_{11} \otimes c_{12} \otimes c_{21} \otimes c_{22} \in \sum_s C_0 \otimes C_1 \otimes D^s \otimes D^s,$$

then

$$\varepsilon_i(c_{11})\varepsilon_j(c_{22})c_{12} \otimes c_{21} \in C_1 \otimes D^j;$$

if $c_{11} \otimes c_{12} \otimes c_{21} \otimes c_{22} \in \sum_{t,k} D^t \otimes D^t \otimes C_1 \otimes D^k$, then

$$\varepsilon_i(c_{11})\varepsilon_j(c_{22})c_{12} \otimes c_{21} \in D^i \otimes C_1;$$

if $c_{11} \otimes c_{12} \otimes c_{21} \otimes c_{22} \in \sum_{t,k} D^t \otimes D^t \otimes D^k \otimes C_1$, then

$$\varepsilon_i(c_{11})\varepsilon_j(c_{22})c_{12} \otimes c_{21} \in D^i \otimes C_0.$$

Thus, in all the cases we have

$$\varepsilon_i(c_{11})\varepsilon_j(c_{22})c_{12} \otimes c_{21} \in D^i \otimes C + C \otimes D^j.$$

This proves $\varepsilon_j \rightharpoonup c \leftharpoonup \varepsilon_i \in D^i \wedge_C D^j$.

(ii) This is straightforward (or, follows from Lemma 2.3.1 of ref. [22]).

(iii) and (iv) Since $(D^i \wedge_C D^j) \cap C_0 = D^i + D^j$ by (ii), it follows that there is a coalgebra embedding

$$(D^i \wedge_C D^j)/(D^i + D^j) \cong ((D^i \wedge_C D^j) + C_0)/C_0 \hookrightarrow C_1/C_0.$$

By the construction of C_0-C_0-bicomodule structure maps ρ_l and ρ_r of C_1/C_0, one observes that

$$(D^i \wedge_C D^j)/(D^i + D^j) \hookrightarrow {}^i(C_1/C_0)^j.$$

It follows from (i) that

$$\begin{aligned} C_1/C_0 &= \left(\sum_{i,j\in I}(D^i \wedge_C D^j)\right)\Big/\, C_0 \\ &= \sum_{i,j\in I}((D^i \wedge_C D^j) + C_0)/C_0 \\ &\hookrightarrow \sum_{i,j\in I} {}^i(C_1/C_0)^j \\ &= \bigoplus_{i,j\in I} {}^i(C_1/C_0)^j. \end{aligned}$$

This forces the embedding $((D^i \wedge_C D^j) + C_0)/C_0 \hookrightarrow {}^i(C_1/C_0)^j$ to be an isomorphism, and hence

$$\begin{aligned} C_1/C_0 &= \sum_{i,j\in I}((D^i \wedge_C D^j) + C_0)/C_0 \\ &= \bigoplus_{i,j\in I}((D^i \wedge_C D^j) + C_0)/C_0 \\ &\cong \bigoplus_{i,j\in I}(D^i \wedge_C D^j)/(D^i + D^j). \end{aligned}$$

□

Theorem 4.1 also permits us to slightly modify the definition of the link quiver of a coalgebra, by allowing multiple arrows. Of course, in the case for the basic coalgebras, it is exactly the original definition.

Definition 4.1[4]**.** Let C be a coalgebra. The link quiver of C is defined as follows. The vertices are the isoclasses of simple subcoalgebras of C; and for two simple subcoalgebras D^i and D^j of C, there are

$$l_{ij} := \frac{1}{n_i n_j}\dim_K(D^j \wedge_C D^i)/(D^i + D^j)$$

arrows from i to j, where n_i is the positive integer such that $(D^i)^* \simeq M_{n_i}(\Delta_i)$, where Δ_i is a finite-dimensional division algebra over K.

Corollary 4.1. The link quiver of coalgebra C coincides with the Gabriel quiver of C.

Proof. It follows from Theorem 4.1(iv) that

$$\begin{aligned} l_{ij} &= \frac{1}{n_i n_j}\dim_K(D^j \wedge_C D^i)/(D^i + D^j) \\ &= \frac{1}{n_i n_j}\dim_K {}^j(C_1/C_0)^i \\ &= \dim_K e_i.{}^j(C_1/C_0)^i.e_j \\ &= t_{ij}. \end{aligned}$$

□

5 Locally finite coalgebras

We give a new characterization of locally finite coalgebras, as an application of Theorems 3.1 and 4.1.

By definition, a coalgebra C is said to be locally finite, provided that the wedge $V \wedge_C W$ is finite-dimensional whenever V and W are both finite-dimensional. By the fundamental theorem on coalgebras (i.e. each finite-dimensional subspace of a coalgebra is contained in a finite-dimensional subcoalgebra), it is clear that a coalgebra C is locally finite if and only if $D \wedge_C D$ is finite-dimensional for each finite-dimensional subcoalgebra D of C.

Heyneman-Radford showed that a reflexive coalgebra is locally finite (see 3.2.4 of ref. [22]); Conversely, if C is locally finite with C_0 being finite-dimensional, then C is reflexive (see 4.2.6 of ref. [22]).

Recall that a subcoalgebra D of C is said to be saturated provided that $D \wedge_C D = D$.

Let $C = C' \oplus C''$ as coalgebras and let (M, ρ_l, ρ_r) be a C-C-bicomodule. Set

$$N = \{m \in M \mid \rho_l(m) \in C' \otimes M, \quad \rho_r(m) \in M \otimes C'\}.$$

Then N is a C'-C'-bicomodule.

Lemma 5.1. With the above notation, $\mathrm{Cot}_{C'}(N)$ is a saturated subcoalgebra of $\mathrm{Cot}_C(M)$.

Proof. Set $\widetilde{C} := \mathrm{Cot}_C(M)$. By the construction of $\mathrm{Cot}_{C'}(N)$ we have

$$\mathrm{Cot}_{C'}(N) = \bigcup_{n\geqslant 1} \wedge^n_{\widetilde{C}} C',$$

where $\wedge^n_{\widetilde{C}} C' = C' \wedge_{\widetilde{C}} C' \wedge_{\widetilde{C}} \cdots \wedge_{\widetilde{C}} C'$ (n times). Hence $\mathrm{Cot}_{C'}(N)$ is saturated in $\widetilde{C}$ (see 2.1.1 of ref. [22]). $\square$

The main result of this section is

Theorem 5.1. The Gabriel quiver of a locally finite coalgebra C is locally finite (i.e., there are only finitely many arrows between arbitrary two vertices).

Conversely, if the Gabriel quiver of C is locally finite and C_0 is separable, then C is locally finite.

Proof. The necessity follows form Corollary 4.1 since the simple coalgebras are finite-dimensional by the fundamental theorem of coalgebras.

Conversely, assume that the Gabriel quiver of C is locally finite and C_0 is separable. In order to prove that C is locally finite, by Theorem 3.1 it suffices to show that the cotensor coalgebra $\mathrm{Cot}_{C_0}(C_1/C_0)$ is locally finite. This is because a subcoalgebra of a locally finite coalgebra is again locally finite (see 2.3.2 of ref. [22]). In the following, we denote $\mathrm{Cot}_{C_0}(C_1/C_0)$ by $\widetilde{C}$.

Let D be an arbitrary finite-dimensional subcoalgebra of $\widetilde{C}$. Then the coradical D_0 of D is a direct summand of C_0. Set

$$M := \{x \in C_1/C_0 \mid \rho_l(x) \in D_0 \otimes C_1/C_0, \quad \rho_r(x) \in C_1/C_0 \otimes D_0\}.$$

Note that D_0 is finite-dimensional and M is contained in a direct sum of finitely many ${}^i(C_1/C_0)^j$'s. Since the Gabriel quiver of C is locally finite, it follows that each ${}^i(C_1/C_0)^j$ is finite-dimensional, and hence M is finite-dimensional. By Theorem 3.1 we have

$$D \subseteq \mathrm{Cot}_{D_0}(D_1/D_0) \subseteq \mathrm{Cot}_{D_0}(M).$$

By Lemma 5.1 $\mathrm{Cot}_{D_0}(M)$ is a saturated subcoalgbra of $\widetilde{C}$. It follows that

$$D \wedge_{\widetilde{C}} D \subseteq \mathrm{Cot}_{D_0}(M) \wedge_{\widetilde{C}} \mathrm{Cot}_{D_0}(M) = \mathrm{Cot}_{D_0}(M).$$

Since D is of finite dimension, we may assume that $D \subseteq \oplus_{i\leqslant n} M^{\Box i}$ for some n. It follows that $D \wedge_{\widetilde{C}} D$ is contained in $\oplus_{i\leqslant 2n} M^{\Box i}$ (see Remark 1.1), which is also of finite dimension. This proves that the cotensor coalgebra $\mathrm{Cot}_{C_0}(C_1/C_0)$ is locally finite. $\square$

6 Quasi-coFrobenius coalgebras

Recall that a coalgebra C is said to be left quasi-coFrobenius if there exists an injective C^*-module map from C to a free left C^*-module, where the left C^*-module structure on C is given as in 4.2. Similarly, one has the concept of the right quasi-coFrobenius coalgebras. A coalgebra is quasi-coFrobenius if it is both left quasi-coFrobenius and right quasi-coFrobenius. Note that a coalgebra C is left quasi-coFrobenius if and only if every injective right C-comodule is projective (see Theorem 3.3.4 in ref. [16]); and that if C is left quasi-coFrobenius, then C^* is right quasi-Frobenius (see Corollary 3.3.9 in ref. [16]). Also note that if C is finite-dimensional, then C is left quasi-coFrobenius if and only if C is right quasi-coFrobenius, if and only if C^* is quasi-Frobenius.

We need the following fact, which seems to be well-known.

Lemma 6.1. Let C be a coalgebra. Then C is indecomposable if and only if the dual algebra C^* is indecomposable.

Proof. Note that here C is not necessarily finite-dimensional. The "if" part is trivial. It suffices to prove the "only if " part. If $C^* \simeq A_1 \times A_2$ as algebras, then $C^{*\circ} \simeq A_1^\circ \oplus A_2^\circ$ as coalgebras, where A° denotes the finite dual of an algebra A. Let $\phi : C \longrightarrow C^{**}$ be the natural embedding. Then the image of ϕ is contained in $C^{*\circ}$ (see, e.g. Proposition 1.5.12 in ref. [16]). Identify C with $\phi(C)$. Then

$$C \simeq ({A_1}^\circ \cap C) \oplus ({A_2}^\circ \cap C)$$

as coalgebras. Note that $A_i^\circ \cap C \neq \{0\}$ $(i = 1, 2)$ (Otherwise, say $A_1^\circ \cap C = \{0\}$, then $C = A_2^\circ \cap C$, i.e. C is contained in A_2°, it follows that A_1 vanishes on C, and hence $A_1 = 0$). This completes the proof. $\square$

The main result of this section is

Theorem 6.1. Let C be an indecomposable non-simple coalgebra. If C is a left quasi-coFrobenius, then the Gabriel quiver of C has no sources.

Thus, the Gabriel quiver of a non-simple quasi-coFrobenius coalgebra has no sources and no sinks.

Proof. Otherwise, assume that the Gabriel quiver Q of C has a source $i \in I$. Let S_i be the corresponding right simple comodule. Then by Lemma 2.1 we have $\operatorname{soc}(E(S_i)/S_i) \simeq \oplus_{j\in I} \frac{t_{ji}}{d_j} S_j$. Since i is a source, it follows that $t_{ji} = 0$ for every $j \in I$, and hence $E(S_i) = S_i$.

For any $j \neq i$, we have

$$\operatorname{Hom}_C(E(S_i), E(S_j)) = \operatorname{Hom}_C(S_i, E(S_j)) = \operatorname{Hom}_C(S_i, S_j) = 0,$$

here we have used $\operatorname{soc}(E(S_j)) = S_j$ and the Schur lemma.

On the other hand, since C is left quasi-coFrobenius, it follows that $E(S_i) =$

S_i is projective as a right C-comodule. Thus

$$\mathrm{Hom}_C(E(S_j), E(S_i)) = 0$$

for each $j \neq i \in I$ (otherwise, let $f : E(S_j) \longrightarrow E(S_i) = S_i$ be a nonzero C-comodule map. Then f is surjective . Thus $E(S_j) \simeq S_i \oplus \mathrm{Ker}(f)$ by the projectivity of S_i, a contradiction).

Note that C itself is a right C-comudule via Δ_C, and that there is an algebra isomorphism $\mathrm{End}_C(C) \simeq C^*$, sending f to $\varepsilon_C \circ f$ (see, e.g. Proposition 3.1.8 in ref. [16]). Since $C \simeq \oplus_{j\in I} n_j E(S_j)$ as a right C-comodule, it follows that

$$C^* \simeq \mathrm{End}_C(C) \simeq \mathrm{End}_C(n_i E(S_i)) \oplus \mathrm{End}_C(\oplus_{j\neq i}(n_j E(S_j))).$$

While C^* is indecomposable by Lemma 6.1, we then obtain a desired contradiction. □

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